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# On an enumerative algorithm for solving eigenvalue complementarity problems

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**Abstract** In this paper, we discuss the solution of linear and quadratic eigenvalue complementarity problems (EiCPs) using an enumerative algorithm of the type introduced by Júdice et al. [1]. Procedures for computing the interval that contains all the eigenvalues of the linear EiCP are first presented. A nonlinear programming (NLP) model for the quadratic EiCP is formulated next, and a necessary and sufficient condition for a stationary point of the NLP to be a solution of the quadratic EiCP is established. An extension of the enumerative algorithm for the quadratic EiCP is also developed, which solves this problem by computing a global minimum for the NLP formulation. Some computational experience is presented to highlight the efficiency and efficacy of the proposed enumerative algorithm for solving linear and quadratic EiCPs.

**Keywords** Eigenvalue Problems · Complementarity Problems · Nonlinear Programming · Global Optimization

**Mathematics Subject Classification (2000)** 90B60 · 90C33 · 90C30 · 90C26

## 1 Introduction

The *Eigenvalue Complementarity Problem* (**EiCP**) [2,3] consists of finding a real number $\lambda$ and a vector $x \in \mathbb{R}^n \backslash \{0\}$ such that

$$w = (\lambda B - A)x \tag{1}$$

$$w \geq 0, x \geq 0 \tag{2}$$

$$x^T w = 0, \tag{3}$$

where $w \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{n \times n}$, and $B$ is positive definite (PD), i.e., $x^T Bx > 0$ for all $x \neq 0$. This problem arises in applications within different areas of science and engineering [2,4,5]. Since the problem

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is homogeneous, the constraint

$$e^T x = 1 \tag{4}$$

can be added without loss of generality, where $e \in \mathbb{R}^n$ is a vector of ones.

The EiCP is equivalent to the following variational inequality (VI) problem [1]: Find a vector $x \in \Delta$ such that

$$F(x)^T(y-x) \geq 0, \ \forall y \in \Delta, \tag{5}$$

where $F\colon \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n$ is defined by

$$F(x) = \left(\frac{x^T A x}{x^T B x} B - A\right) x, \tag{6}$$

and the set $\Delta$ is the unit simplex in $\mathbb{R}^n$, i.e.,

$$\Delta = \{x \in \mathbb{R}^n \colon e^T x = 1, \ x \geq 0\}. \tag{7}$$

As the VI on the simplex has a solution, the same is true for the EiCP. Therefore the EiCP always has a solution when $B$ is PD, which can be computed by finding a solution $\bar{x}$ to the VI and then setting $\bar{\lambda} = \frac{\bar{x}^T A \bar{x}}{\bar{x}^T B \bar{x}}$. As described in [6], the VI can be reformulated as an optimization problem involving the minimization of a regularized gap-function. A hybrid algorithm has been proposed in [7] for solving the EiCP by computing a stationary point of this last function. The algorithm incorporates projection techniques and the so-called modified Josephy-Newton method and seems to work well in general, but may fail to find a solution for the EiCP in some instances. Another projection method that deals with the original formulation of the EiCP has been introduced in [8]. The EiCP can also be formulated as a nonsmooth system of nonlinear equations based on the so-called NCP-functions, and then solved by a semi-smooth Newton's method, as discussed in [9]. This approach seems to be more robust than the projection method, but both fail to find a solution to the EiCP in some instances. DCA algorithms have also been proposed to deal with the EiCP [10] with similar conclusions. Finally, the EiCP can be posed as a Nonlinear Complementarity Problem (NCP) [2] and then solved by a path-following algorithm [6], such as PATH [11], or by an interior-point method [12], such as LOQO [13]. Again, these methods are not always able to find a solution to the EiCP. It is also important to add that if $A$ and $B$ are symmetric, then the EiCP is much easier to solve, as it reduces to finding a stationary point of an appropriate merit function on the simplex [14–17].

Recognizing the inability of nonlinear optimization algorithms to solve the EiCP in all cases, an enumerative method was introduced in [18], and subsequently improved in [1]. This method computes a solution to the EiCP by searching for a global minimum of the following nonlinear programming (NLP) formulation of the EiCP [18, 1]:

$$\begin{aligned}
&\text{Minimize } \|y - \lambda x\|_2^2 + x^T w && (8)\\
&\text{subject to } w = By - Ax && (9)\\
&\qquad e^T x = 1 && (10)\\
&\qquad e^T y = \lambda && (11)\\
&\qquad w \geq 0, x \geq 0 && (12)\\
&\qquad l \leq \lambda \leq u, && (13)
\end{aligned}$$

where $l$ and $u$ are end-points of an interval containing at least one eigenvalue. The algorithm uses two branching strategies, the first based on the dichotomy of the complementary variables ($x_i = 0$ or $w_i = 0$) and the other consisting of bracketing the interval $[l, u]$. However, no pratical indication was provided on how to compute the values of $l$ and $u$. This is the first motivation of the present paper.

A Quadratic Eigenvalue Complementarity Problem (QEiCP) was recently introduced in [19], motivated by applications mentioned in [19]. This problem consists of finding $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n \backslash \{0\}$ such that

$$\begin{aligned}
w &= \lambda^2 Ax + \lambda Bx + Cx && (14)\\
w &\geq 0, x \geq 0 && (15)\\
x^T w &= 0 && (16)\\
e^T x &= 1, && (17)
\end{aligned}$$

where $w \in \mathbb{R}^n$ and $A, B, C \in \mathbb{R}^{n\times n}$. In contrast with the linear EiCP, the problem may have no solution even when the matrix $A$ of the leading term is PD. By recognizing this fact, the concepts of co-regularity and co-hiperbolicity were described in [19], where it was shown that the QEiCP can be reduced to a VI on the simplex when both the aforementioned properties are satisfied and therefore has a solution. In this paper, we discuss an extension of the enumerative method [1] for the solution of the QEiCP when these two properties hold. As before, the algorithm aims to find a global minimum of an appropriate nonlinear programming formulation of the QEiCP. Procedures for computing $l$ and $u$ are first discussed. The algorithm is based on the branching strategies mentioned above for the EiCP and is always able to find a solution to the QEiCP. Numerical results reported in this paper reveal that the algorithm works well in practice, similar to the EiCP, and appears to be more efficient than the well-known and robust global optimization code BARON [20] for the solution of the same instances.

The structure of the remainder of this paper is as follows. In Section 2, a brief review of the enumerative method for the EiCP is presented. The new procedures for computing the end-points of the interval $[l, u]$ containing all the eigenvalues are discussed in Section 3. The Quadratic Eigenvalue Complementarity Problem and its nonlinear programming formulation are introduced in Section 4. Implementation aspects for the computation of the end-points of the interval $[l, u]$ and the enumerative method for the QEiCP are discussed in Sections 5 and 6, respectively. Computational experiments concerned with solving EiCPs and QEiCPs using the enumerative algorithm are reported in Section 7, and some concluding remarks are provided in the final section of this paper.

## 2 An enumerative algorithm for the EiCP

By eliminating the $w-$variables from the definition of the NLP (8)-(12), (and tentatively dropping the imposed bounds on $\lambda$), we derive the following equivalent nonlinear program:

$$\begin{aligned}
\textbf{P1: } \text{Minimize } & f(x, y, \lambda) = (y - \lambda x)^T (y - \lambda x) + x^T (By - Ax) \\
\text{subject to } & By - Ax \geq 0 \\
& e^T x = 1 \\
& e^T y = \lambda \\
& x \geq 0.
\end{aligned}$$

Then the following result holds:

**Theorem 1** *[18] The EiCP has a solution $(\bar{x}, \bar{\lambda})$ if and only if Problem P1 has a global minimum $(\bar{x}, \bar{y}, \bar{\lambda})$ with zero optimal value.*

Finding a stationary point of this NLP is a much easier task and it is therefore important to investigate whether such a point is a solution to the EiCP. The following theorem provides a necessary and sufficient condition for such a result to hold.

**Theorem 2** *A stationary point $(\bar{x}, \bar{y}, \bar{\lambda})$ of Problem P1 is a solution to the EiCP if and only if $\alpha_1 = 0$ and $\alpha_2 = 0$, where $\alpha_1$ and $\alpha_2$ are the Lagrange multipliers associated with the linear equalities $e^T x = 1$ and $e^T y - \lambda = 0$, respectively.*

The proof of this Theorem is similar to that of Theorem 3.2 of [1], and shows that the stated conditions on the Lagrange multipliers at a stationary point are equivalent to having a zero objective value at such a solution for Problem P1. Theorem 2 is more general than the latter result in [1] since $\lambda$ is not assumed to be positive, and reduces to it under such a hypothesis.

It follows from Theorem 2 that a stationary point for Problem P1 might not in general solve the EiCP. By recognizing this fact, an enumerative method was introduced in [1] to guarantee finding such a solution by computing stationary points of the objective function of Problem P1 in a systematic way until a solution to the EiCP is detected via a stationary point having an objective function value of zero. For this purpose, a compact interval $[l, u]$ for the variable $\lambda$, which contains at least one eigenvalue, was imposed in order to facilitate the search process and to establish the global convergence of the algorithm to a solution of the EiCP. In the next section we discuss some novel techniques for computing $l$ and $u$. Note that Theorem 2 also holds when $\bar{\lambda} \in (l, u)$, which can be assumed without loss of generality, given the foregoing bound computations.

Assuming that $[l, u]$ has been computed, and incorporated within Problem P1, the aforementioned enumerative algorithm explores a binary tree that is constructed under two branching strategies, namely, based on a pair of positive complementary variables at the current stationary point of the Problem P1 and by partitioning the interval $[l, u]$. Therefore, each node $k$ of the enumeration tree is associated with an interval $[\bar{l}, \bar{u}] \subseteq [l, u]$ along with two sets $I$ and $J$ that respectively, record those $w-$ and $x-$variables that are presently fixed to zero. Since $y_i = \lambda x_i$, $i = 1, \ldots, n$, in any solution to the EiCP, the following constraints are thus associated with node $k$ of the tree:

$$\begin{gathered}\bar{l}x_i \leq y_i \leq \bar{u}x_i, \quad \forall i \in \bar{J} \\ y_i = x_i = 0, \quad \forall i \in J \\ w_i = 0, \quad \forall i \in I,\end{gathered}$$

where $l \leq \bar{l} < \bar{u} \leq u$, $J \subseteq \{1, \ldots, n\}$, $\bar{J} = \{1, \ldots, n\} \backslash J$, and $J \cap I = \emptyset$. Furthermore, consider the sets

$$K = I \cup J, \ \bar{I} = \{1, \ldots, n\} \backslash I, \text{ and } \bar{K} = \{1, \ldots, n\} \backslash K.$$

Then the subproblem at node $k$ is given as follows, where any set-subscript on a variable restricts the variable indices to the corresponding set:

$$\begin{aligned}\textbf{P1}(k)\textbf{:}\quad \text{Minimize } & f(x, y, w, \lambda) = (y_{\bar{J}} - \lambda x_{\bar{J}})^T (y_{\bar{J}} - \lambda x_{\bar{J}}) + x_{\bar{K}}^T w_{\bar{K}} \\ \text{subject to } & w = By - Ax \\ & e^T x_{\bar{J}} = 1 \\ & e^T y_{\bar{J}} = \lambda \\ & \bar{l} \leq \lambda \leq \bar{u} \\ & \bar{l}x_j \leq y_j \leq \bar{u}x_j\,, \quad \forall j \in \bar{J} \\ & w_{\bar{I}} \geq 0, x_{\bar{J}} \geq 0 \\ & y_j = x_j = 0, \quad \forall j \in J \\ & w_i = 0, \quad \forall i \in I.\end{aligned}$$

At this node $k$, the algorithm searches for a stationary point to the corresponding program P1($k$). If the objective function value at this stationary point is zero, then a solution to the EiCP is at hand and the algorithm terminates. Otherwise, two new nodes are created and the process is repeated. The algorithm also includes heuristic rules for choosing an open node from some associated list and for deciding which of the two branching strategies should be used at the selected node $k$ whenever a stationary point having a positive objective function value is found for P1($k$). The formal steps of the algorithm are presented below, where the cases of EiCP not having a solution would not arise under our stated assumptions.

**Enumerative algorithm**

**Step 0 (Initialization) -** Let $\epsilon_1$, and $\epsilon_2$ be selected tolerances, where $0 < \epsilon_1 < \epsilon_2$ (we can take $\epsilon_1 = \epsilon^2$ and $\epsilon_2 = \epsilon$ for some $0 < \epsilon < 1$, for example). Set $k = 1$, $I = \emptyset$, $J = \emptyset$, and find a stationary point $(\bar{x}, \bar{y}, \bar{w}, \bar{\lambda})$ of P1(1). If P1(1) is infeasible, then EiCP has no solution; terminate. Otherwise, let $L = \{1\}$ be the set of open nodes, set $UB(1) = f(\bar{x}, \bar{y}, \bar{w}, \bar{\lambda})$, and let $N = 1$ be the number of nodes generated.

**Step 1 (Choice of node) -** If $L = \emptyset$ terminate; EiCP has no solution. Otherwise, select $k \in L$ such that
$$UB(k) = \min\{UB(i) \colon i \in L\},$$

and let $(\bar{x}, \bar{y}, \bar{w}, \bar{\lambda})$ be the stationary point that was previously found at this node.

**Step 2 (Branching rule) -** Let

$$\theta_1 = \max\{\bar{w}_i \bar{x}_i : i \in \bar{K}\} = \bar{w}_r \bar{x}_r, \text{ and}$$
$$\theta_2 = \max\{|\bar{y}_i - \bar{\lambda}\bar{x}_i| : i \in \bar{J}\}\ .$$

(i) If $\theta_1 \leq \epsilon_1$ and $\theta_2 \leq \epsilon_2$ then $\bar{\lambda}$ yields a complementary eigenvalue (within the tolerance $\epsilon_2$) with $\bar{x}$ being a corresponding eigenvector; terminate.
(ii) If $\theta_1 > \theta_2$, branch on the complementary variables $(w_r, x_r)$ associated with $\theta_1$ and generate two new nodes, $N + 1$ and $N + 2$.
(iii) If $\theta_1 \leq \theta_2$, then partition the interval $[\bar{l}, \bar{u}]$ at node $k$ into $[\bar{l}, \tilde{\lambda}]$ and $[\tilde{\lambda}, \bar{u}]$ to generate two new nodes, $N + 1$ and $N + 2$, where

$$\tilde{\lambda} = \begin{cases} \bar{\lambda} & \text{if } \min\{(\bar{\lambda} - \bar{l}), (\bar{u} - \bar{\lambda})\} \geq 0.1(\bar{u} - \bar{l}) \\ \frac{\bar{u}+\bar{l}}{2} & \text{otherwise.} \end{cases}$$

**Step 3 (Solve, Update, and Queue) -** For each of $t = N + 1$ and $t = N + 2$, find a stationary point $(\tilde{x}, \tilde{y}, \tilde{w}, \tilde{\lambda})$ of Problem P1($t$). If P1($t$) is feasible, set $L = L \cup \{t\}$ and $UB(t) = f(\tilde{x}, \tilde{y}, \tilde{w}, \tilde{\lambda})$. Set $L = L \backslash \{k\}$ and return to Step 1.

The convergence of this enumerative method follows from Theorem 4.1 of [1]. Note that the algorithm finds a global minimum of Problem P1. If the interval $[l, u]$ contains at least one eigenvalue as assumed, then the objective function value at the global minimum is zero and provides a solution to the EiCP. Hence, the objective function value is positive at the end of the algorithm if and only if the EiCP has no eigenvalue in the interval $[l, u]$.

## 3 Finding lower and upper bounds for the eigenvalues

### 3.1 Finding an upper bound

We start by discussing two procedures for computing an upper bound $u$ for the variable $\lambda$. In the first technique, we consider the EiCP with $B$ taken as the identity matrix. The following theorem derives a formula for computing a value for $u$.

**Theorem 3** *Every eigenvalue $\lambda$ for the EiCP satisfies*

$$|\lambda| \leq \min\{\|A\|_1, \|A\|_\infty\},$$

*where* $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$ *and* $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$.

*Proof* If $\lambda$ is an eigenvalue for EiCP, then $\lambda$ is an eigenvalue for a principal submatrix $A_{II}$ of $A$, where $I \subseteq \{1, \ldots, n\}$. Let $\rho(A)$ denote the spectral radius of the matrix $A$, i. e.,

$$\rho(A) = \max |\lambda_i|\ ,$$

where $\lambda_i$ are the real eigenvalues of $A$. Then [21]

$$\rho(A) \leq \min\{\|A\|_1,\ \|A\|_\infty\}\ ,$$

where $\|A\|_1$ and $\|A\|_\infty$ represent the $l_1$ and $l_\infty$ norms of $A$ [21], respectively. Now, if $\lambda_i$ denotes the eigenvalues of $A_{II}$, then

$$\rho(A_{II}) = \max |\lambda_i| \leq \min\{\|A_{II}\|_1,\ \|A_{II}\|_\infty\} \leq \min\{\|A\|_1,\ \|A\|_\infty\}\ .$$

□

So, an upper bound $u$ can be computed as follows

$$u = \min\{\|A\|_1, \|A\|_\infty\}.$$

A drawback of this approach is the need for $B$ to be the identity matrix. The next result provides a second procedure for computing $u$ in the general case.

**Theorem 4** *Let $c_i = \max\{a_{ij}, j = 1, \ldots, n\}$, $d_i = \max\{c_i, 0\}$ for all $i = 1, \ldots, n$, and $d \in \mathbb{R}^n$ be a vector with components $d_i$. Then we can take*

$$u = \frac{d^T \bar{x}}{\bar{x}^T B \bar{x}}\,,$$

*where $\bar{x}$ is a stationary point of $\max\{\frac{d^T x}{x^T B x} : x \in \Delta\}$.*

*Proof* If $\lambda$ is a solution of EiCP, then [1]

$$\exists x \in \Delta\colon \quad \lambda = \frac{x^T A x}{x^T B x}.$$

But

$$x^T A x = \sum_{i=1}^{n} x_i (Ax)_i \leq \sum_{i=1}^{n} c_i x_i$$

where

$$c_i \equiv \max_{x \in \Delta} \sum_{j=1}^{n} a_{ij} x_j = \max\{a_{ij}\,,\ j = 1, \ldots, n\}\,, \quad i = 1, \ldots, n. \tag{18}$$

Now, consider the function

$$g(x) = \frac{d^T x}{x^T B x}\,,$$

where

$$d_i = \max\{c_i, 0\}\,, \quad i = 1, \ldots, n.$$

Since the function in the numerator is nonnegative and concave over $\Delta$ and the denominator function is strictly convex, then $g$ is explicitly quasi-concave on $\Delta$ [22] and any stationary point of

$$\max_{x \in \Delta} g(x)$$

is a global maximum of $g$ at $\Delta$. Moreover, by the linearity and compactness of $\Delta$, such a stationary point exists. Since $\lambda \leq \max_{x \in \Delta} g(x)$, this completes the proof. $\square$

### 3.2 Finding a lower bound

It follows from Theorem 3 that a lower bound $l$ can be computed by

$$l = -\min\{\|A\|_1, \|A\|_\infty\}.$$

As before, this procedure requires $B$ to be the identity matrix. Next, we introduce a new procedure for computing $l$ in the general case. Let $u$ be an upper bound for the variable $\lambda$ that has been computed as explained in Subsection 3.1. Furthermore, let

$$\bar{u} = \max\{0, u\}. \tag{19}$$

Since $y = \lambda x$ in any solution to the EiCP, and $0 \leq x_i \leq 1$ for all $i = 1, \ldots, n$, then $y \leq \bar{u}e$. Now, consider the following linear program (LP):

$$\begin{aligned} &\text{Minimize } \lambda \\ &\text{subject to } w = By - Ax \\ &\qquad e^T x = 1 \\ &\qquad e^T y = \lambda \\ &\qquad w \geq 0, x \geq 0 \\ &\qquad y \leq \bar{u}e, \end{aligned}$$

which is equivalent to

$$\textbf{P2:} \text{ Minimize } e^T y \tag{20}$$
$$\text{subject to } By - Ax \geq 0 \tag{21}$$
$$e^T x = 1 \tag{22}$$
$$x \geq 0 \tag{23}$$
$$y \leq \bar{u}e. \tag{24}$$

An optimal solution to Problem P2, if it exists, provides the required lower bound $l$. The next result shows that such a value indeed exists.

**Theorem 5** *Problem P2 has an optimal solution.*

*Proof* Since EiCP always has a solution under our assumptions, then Problem P2 is feasible. So, it remains to show that Problem P2 has no nonzero recession direction $d = \begin{bmatrix} d_x \\ d_y \end{bmatrix}$, where $d_x$ and $d_y$ are the components of $d$ corresponding to the $x-$ and $y-$variables, respectively [23]. From (21)-(24), any such recession direction must satisfy $d_x = 0$, $Bd_y \geq 0$, and $d_y \leq 0$, which implies that $d_y^T B d_y \leq 0$, or that $d_y = 0$ since $B$ is a PD matrix. Hence, P2 has no recession direction. □

*Remark 1* We could also design procedures for finding lower and upper bounds $l$ and $u$ based on the computation of generalized eigenvalues (as $B$ may not be the identity matrix). In our experience, the bounds proposed above are easily computed and are reasonably tight enough to yield an effective algorithmic performance, and so the additional computational effort of such alternative approaches might not benefit the overall efficiency of the algorithm. Nevertheless, it is of interest to study the relative tightness of the bounds derived using such methods (including those based on classical eigenvalues of symmetrized versions of the matrices $A$ and $B$), and to compare their effect on the overall algorithmic procedure. We propose this investigation for future research.

## 4 The quadratic eigenvalue complementarity problem (QEiCP)

The QEiCP, which was introduced in [19], consists of finding a real number $\lambda$ and a vector $x \in \mathbb{R}^n \backslash \{0\}$ such that

$$w = \lambda^2 Ax + \lambda Bx + Cx \tag{25}$$
$$w \geq 0, x \geq 0 \tag{26}$$
$$x^T w = 0, \tag{27}$$

where $A, B, C \in \mathbb{R}^{n \times n}$ are given matrices. As for the EiCP, we can add the constraint $e^T x = 1$ without loss of generality. Compared with the EiCP, there exists an additional term involving $\lambda^2$ in QEiCP. Contrary to the EiCP, the QEiCP may have no solution even when the matrix $A$ of the leading term is PD. For instance, if $B = 0$, and $A$ and $C$ are PD, there is no $0 \neq x \geq 0$ such that $x^T w = 0$. On the other hand, consider the triplet $\Gamma$ of matrices $(A, B, C)$ satisfying the so-called *co-regular* and *co-hyperbolic* conditions [19]:

(i) $x^T Ax \neq 0$ for all $0 \neq x \geq 0$ .
(ii) $(x^T Bx)^2 \geq 4(x^T Ax)(x^T Cx)$ for all $0 \neq x \geq 0$ .

For instance, if $A$ is PD and $-C$ is copositive (i.e., $x^T Cx \leq 0$ for all $x \geq 0$), then $(A, B, C) \in \Gamma$.

Since $x^T w = \lambda^2(x^T Ax) + \lambda(x^T Bx) + x^T Cx = 0$ for any solution to QEiCP, then we can write

$$\lambda = \lambda(x) \equiv \frac{-x^T Bx \pm \sqrt{(x^T Bx)^2 - 4(x^T Ax)(x^T Cx)}}{2x^T Ax}. \tag{28}$$

As shown in [19], QEiCP reduces to a VI on the simplex, whence the QEiCP has a solution whenever $(A, B, C) \in \Gamma$. In Section 6, we describe an extension of the enumerative method discussed in Section 2 for the solution of QEiCP when $(A, B, C) \in \Gamma$. We further assume that $A$ is PD. Since there are two terms respectively involving $\lambda$ and $\lambda^2$, it is natural to introduce two new corresponding vectors $y = \lambda x$ and $z = \lambda y = \lambda^2 x$. As before, we assume that $\lambda$ belongs to a compact interval $[l, u]$ containing all the solutions of the QEiCP, since, as shown in the next section, we can compute finite end-points of such an interval. Because $e^T x = 1$, $x \geq 0$, and $z = \lambda^2 x$, then $z \geq 0$ and $e^T z \leq p$, where $p \equiv \max_{\lambda \in [l,u]} \lambda^2$. So it is natural to introduce the following NLP formulation of the QEiCP:

$$\begin{aligned}
\textbf{P3:}\ \text{Minimize}\ & f(x, y, w, z, \lambda) = (y - \lambda x)^T (y - \lambda x) + (z - \lambda y)^T (z - \lambda y) + x^T w \\
\text{subject to}\ & w = Az + By + Cx \\
& e^T x = 1 \\
& e^T y = \lambda \\
& e^T z \leq p \\
& w \geq 0, x \geq 0, z \geq 0\,.
\end{aligned} \tag{29}$$

The following result holds trivially:

**Theorem 6** *QEiCP has a solution $(\bar{x}, \bar{\lambda})$ if and only if Problem P3 has a global minimum $(\bar{x}, \bar{y}, \bar{w}, \bar{z}, \bar{\lambda})$ with zero optimal value.*

As for the EiCP, it is natural to ask whether a stationary point of this NLP solves the QEiCP. The following result answers this question:

**Theorem 7** *A stationary point $(\bar{x}, \bar{y}, \bar{w}, \bar{z}, \bar{\lambda})$ of Problem P3 is a solution to the QEiCP if and only if $\alpha_2 \bar{\lambda} = 0$ and $\alpha_1 + \alpha_3 p = 0$, where $\alpha_i$, $i = 1, 2, 3$, are the Lagrange multipliers associated with the linear equalities $e^T x = 1$, $e^T y - \lambda = 0$, and $e^T z \leq p$, respectively.*

*Proof* Let $(x, y, w, z, \lambda)$ be a stationary point of Problem P3. We can eliminate the $w_i-$variables from P3 to get the following equivalent program:

$$\begin{aligned}
\text{Minimize}\ & f(x, y, z, \lambda) = (y - \lambda x)^T (y - \lambda x) + (z - \lambda y)^T (z - \lambda y) + x^T (Az + By + Cx) \\
\text{subject to}\ & Az + By + Cx \geq 0 \quad (v \geq 0) \\
& e^T x - 1 = 0 \quad (\alpha_1) \\
& e^T y - \lambda = 0 \quad (\alpha_2) \\
& p - e^T z \geq 0 \quad (\alpha_3 \geq 0) \\
& x \geq 0 \quad (\beta \geq 0) \\
& z \geq 0 \quad (\eta \geq 0),
\end{aligned} \tag{30}$$

where the corresponding dual multipliers are specified within brackets alongside each of the constraints. Note that the Lagrange multipliers associated with the constraints $e^T x = 1$, $e^T y = \lambda$ and $e^T z \leq p$ are the same as those associated with these corresponding constraints in P3. The stationary point of this problem satisfies the following KKT conditions, in addition to the constraints (30):

$$\begin{aligned}
&\nabla_x f(x,y,z,\lambda) - C^T v - \alpha_1 e - \beta = 0\\
&\nabla_y f(x,y,z,\lambda) - B^T v - \alpha_2 e = 0\\
&\nabla_z f(x,y,z,\lambda) - A^T v + \alpha_3 e - \eta = 0\\
&\nabla_\lambda f(x,y,z,\lambda) + \alpha_2 = 0\\
&v^T(Az + By + Cx) = 0\\
&\alpha_3(p - e^T z) = 0\\
&\beta^T x = 0\\
&\eta^T z = 0\\
&v \geq 0,\ \beta \geq 0,\ \eta \geq 0,\ \alpha_3 \geq 0,
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
\nabla_x f(x,y,z,\lambda) &= -2\lambda(y-\lambda x) + Az + By + Cx + C^T x\\
\nabla_y f(x,y,z,\lambda) &= 2(y-\lambda x) - 2\lambda(z-\lambda y) + B^T x\\
\nabla_z f(x,y,z,\lambda) &= 2(z-\lambda y) + A^T x\\
\nabla_\lambda f(x,y,z,\lambda) &= -2x^T(y-\lambda x) - 2y^T(z-\lambda y).
\end{aligned}$$

Hence, the following equalities hold:

$$\begin{aligned}
&-2\lambda(y-\lambda x) + Az + By + Cx + C^T x - C^T v - \alpha_1 e - \beta = 0\\
&2(y-\lambda x) - 2\lambda(z-\lambda y) + B^T x - B^T v - \alpha_2 e = 0\\
&2(z-\lambda y) + A^T x - A^T v + \alpha_3 e - \eta = 0\\
&-2x^T(y-\lambda x) - 2y^T(z-\lambda y) + \alpha_2 = 0.
\end{aligned}$$

Multiplying both sides of these equalities by $x^T$, $y^T$, $z^T$, and $\lambda$, respectively, we obtain

$$-2\lambda x^T(y-\lambda x) + x^T Az + x^T By + x^T Cx + x^T C^T x - x^T C^T v - \alpha_1 x^T e = 0 \tag{32}$$
$$2y^T(y-\lambda x) - 2\lambda y^T(z-\lambda y) + y^T B^T x - y^T B^T v - \alpha_2 y^T e = 0 \tag{33}$$
$$2z^T(z-\lambda y) + z^T A^T x - z^T A^T v - \alpha_3 z^T e = 0 \tag{34}$$
$$-2\lambda x^T(y-\lambda x) - 2\lambda y^T(z-\lambda y) + \alpha_2\lambda = 0. \tag{35}$$

Now, adding the equalities (32), (33), and (34), we have

$$\begin{aligned}
&2(y-\lambda x)^T(y-\lambda x) + 2(z-\lambda y)^T(z-\lambda y) + 2x^T(Az+By+Cx) - v^T(Az+By+Cx) -\\
&-\alpha_1 x^T e - \alpha_2 y^T e - \alpha_3 z^T e = 0.
\end{aligned}$$

Since $e^T x = 1$, $e^T y = \lambda$, and $\alpha_3 e^T z = \alpha_3 p$, we get

$$\begin{aligned}
&2(y-\lambda x)^T(y-\lambda x) + 2(z-\lambda y)^T(z-\lambda y) + 2x^T(Az+By+Cx) - v^T(Az+By+Cx) -\\
&-\alpha_1 - \alpha_2\lambda - \alpha_3 p = 0.
\end{aligned}$$

Finally, due to (31), we obtain

$$\begin{aligned}
&2(y-\lambda x)^T(y-\lambda x) + 2(z-\lambda y)^T(z-\lambda y) + 2x^T(Az+By+Cx) =\\
&= \alpha_1 + \alpha_2\lambda + \alpha_3 p.
\end{aligned} \tag{36}$$

If $\alpha_2\lambda = 0$ and $\alpha_1 + \alpha_3 p = 0$, then the objective function value is zero, which means that the stationary point is a solution of the QEiCP. Conversely, suppose that the stationary point $(x, y, w, z, \lambda)$ of Problem P3 is a solution to the QEiCP. Then $f(x, y, w, z, \lambda) = 0$ by Theorem 6. Therefore, Equation (35) implies that

$$\alpha_2\lambda = 2\lambda x^T(y - \lambda x) + 2\lambda y^T(z - \lambda y) = 0.$$

Furthermore, $\alpha_1 + \alpha_3 p = 0$ by Equation (36). □

Theorem 7 shows that a stationary point of Problem P3 may not in general solve the QEiCP. An extension of the enumerative method discussed in Section 2 can therefore be used to compute a solution to the QEiCP by finding a global minimum of P3. As before, the algorithm requires a compact interval $[l, u]$ containing all the solutions $\lambda$ of the QEiCP. In the next two sections, we describe procedures for computing the finite end-points of such an interval and design an extension of the enumerative method for solving QEiCP. Note that, similar to Theorem 2, the result of Theorem 7 also holds with $\lambda$ restricted to the interval $[l, u]$, where, without loss of generality, we can assume that the eigenvalue for any solution to QEiCP belongs to the open interval $(l, u)$.

## 5 Finding lower and upper bounds for the eigenvalues of QEiCP

Consider $\lambda(x)$ as given by (28), where, as assumed above, $(A, B, C) \in \Gamma$ and $A$ is PD. Then we have that

$$\frac{x^T(-B)x}{2x^TAx} - \sqrt{\left(\frac{x^T(-B)x}{2x^TAx}\right)^2 + \frac{x^T(-C)x}{x^TAx}} \le \lambda(x) \le \frac{x^T(-B)x}{2x^TAx} + \sqrt{\left(\frac{x^T(-B)x}{2x^TAx}\right)^2 + \frac{x^T(-C)x}{x^TAx}}. \quad (37)$$

Now, following the proof of Theorem 4, let $d_1$, $d_2$, and $d_3$ be nonnegative vectors such that

$$-d_2^T x \le \frac{1}{2}x^T(-B)x \le d_1^T x \quad \text{and} \quad x^T(-C)x \le d_3^T x\,, \quad \forall x \in \Delta.$$

Accordingly, we get

$$-u_2 \le \frac{x^T(-B)x}{2x^TAx} \le u_1 \quad \text{and} \quad \frac{x^T(-C)x}{x^TAx} \le u_3\,, \quad \forall x \in \Delta, \quad (38)$$

where $u_i \equiv \max\limits_{x\in\Delta}\{d_i^T x / x^TAx\}$, $\forall i = 1, 2, 3$. Note that the function $d_i^T x/x^TAx$ is explicitly quasi-concave on $\Delta$, $\forall i = 1, 2, 3$, and any stationary point of each one of these three functions on $\Delta$ gives the required value $u_i$, $\forall i = 1, 2, 3$. Hence, we obtain from (37) that

$$-u_2 - \sqrt{\max\{u_1^2, u_2^2\} + u_3} \le \lambda(x) \le u_1 + \sqrt{\max\{u_1^2, u_2^2\} + u_3}\,, \quad \forall x \in \Delta.$$

Consequently, we can take

$$l = -u_2 - \sqrt{\max\{u_1^2, u_2^2\} + u_3} \quad \text{and} \quad u = u_1 + \sqrt{\max\{u_1^2, u_2^2\} + u_3}. \quad (39)$$

Note that if $B$ or $-B$ is copositive, then we can respectively take $u_1 \equiv 0$ or $u_2 \equiv 0$ by (38), and likewise, if $C$ is copositive then we can take $u_3 \equiv 0$.

We can also extend Theorem 5 for possibly computing a better lower bound when $B^T$ is an S-matrix [24], that is, there exists a vector $0 \ne v \ge 0$ such that $B^T v > 0$. To do this, let $p = \max\limits_{\lambda\in[l,u]} \lambda^2$ as before, where $l$ and $u$ have been computed by (39), let $\bar{u} \equiv \max\{0, u\}$, and consider the following linear program:

$$
\begin{aligned}
\textbf{P4:}\ & \text{Minimize } e^T y \\
& \text{subject to } Az + By + Cx \geq 0 \\
& e^T x = 1 \\
& e^T z \leq p \\
& y \leq \bar{u} e \\
& z \geq 0, x \geq 0.
\end{aligned}
$$

**Theorem 8** *If $B^T$ is an S matrix then Problem P4 has an optimal solution.*

*Proof* Since $(A, B, C) \in \Gamma$, then Problem P4 is feasible. Furthermore, similar to the proof of Theorem 5, if $d^T \equiv (d_x^T, d_y^T, d_z^T)$ is a recession direction, then $d_x = 0$, $d_z = 0$, and $d_y$ satisfies $Bd_y \geq 0$ and $d_y \leq 0$. But since $B^T$ is an S matrix, there exists $0 \neq v \geq 0$ such that $B^T v > 0$. Thus $(Bd_y)^T \geq 0$ and $d_y^T B^T v \geq 0$, which, together with $B^T v > 0$ and $d_y \leq 0$, yields $d_y = 0$. Thus Problem P4 is bounded. □

Let $\bar{l}$ be the optimal value of P4. Then we can replace $l \leftarrow \max\{l, \bar{l}\}$.

## 6 An enumerative algorithm for the QEiCP

As for the EiCP, the proposed enumerative method aims to find a global minimum for Problem P3 with the additional constraint $l \leq \lambda \leq u$. The algorithm explores a binary tree that is constructed by using the same branching strategies as discussed for the EiCP version. In order to reduce the overall search in this process, a number of constraints are added to the Problem P3 as described next. Since $y = \lambda x$ in any solution of the QEiCP, then

$$
\begin{aligned}
lx_i &\leq \lambda x_i = y_i \\
ux_i &\geq \lambda x_i = y_i
\end{aligned}
$$

for each $i = 1, \ldots, n$. Likewise, as $z = \lambda y = \lambda^2 x$ in any such solution, we have

$$
\begin{aligned}
rx_i &\leq \lambda^2 x_i = z_i \\
sx_i &\geq \lambda^2 x_i = z_i
\end{aligned}
$$

for each $i = 1, \ldots, n$, where $r = \min\limits_{\lambda \in [l,u]} \lambda^2$ and $s = \max\limits_{\lambda \in [l,u]} \lambda^2$.

We can also include the RLT bound-factor constraints [25] $[lx_j \leq y_j \leq ux_j] * [l \leq \lambda \leq u]$, $\forall j = 1, \ldots, n$, which are linearized under the substitutions $\lambda x_j = y_j$, and $\lambda y_j = z_j$, $\forall j = 1, \ldots, n$, to yield the following:

(i) $[lx_j \leq y_j \leq ux_j] * [(\lambda - l) \geq 0] \Rightarrow lx_j(\lambda - l) \leq y_j(\lambda - l) \leq ux_j(\lambda - l)$
$\Rightarrow ly_j - l^2 x_j \leq z_j - ly_j \leq uy_j - lux_j, \quad \forall j = 1, \ldots, n$

(ii) $[lx_j \leq y_j \leq ux_j] * [(u - \lambda) \geq 0] \Rightarrow lx_j(u - \lambda) \leq y_j(u - \lambda) \leq ux_j(u - \lambda)$
$\Rightarrow lux_j - ly_j \leq uy_j - z_j \leq u^2 x_j - uy_j, \quad \forall j = 1, \ldots, n.$

Another valid constraint is $e^T z = \lambda^2$, which implies the inequality $e^T z \leq s$ and follows from the definition of the $z-$ and $y-$ variables. Since it is useful to keep all the constraints linear, we introduce the variable $\gamma$ defined by $\gamma = \lambda^2$ and include RLT bound-factor constraints $[(\lambda - l)^2]_L \geq 0$, $[(u - \lambda)^2]_L \geq 0$, and $[(u - \lambda)(\lambda - l)]_L \geq 0$, where $[.]_L$ denotes the linearization of $[.]$ under the foregoing substitution process, to derive the following constraints:

(i) $e^T z = \gamma$
(ii) $[(\lambda - l)^2]_L \geq 0 \Rightarrow \gamma - 2l\lambda + l^2 \geq 0$
(iii) $[(u - \lambda)^2]_L \geq 0 \Rightarrow \gamma - 2u\lambda + u^2 \geq 0$
(iv) $[(u - \lambda)(\lambda - l)]_L \geq 0 \Rightarrow (u + l)\lambda - \gamma - ul \geq 0.$

By adding these $(8n + 4)$ constraints, we obtain the following augmented nonlinear program associated with the QEiCP:

**P5:** Minimize $f(x, y, w, z, \lambda) = (y - \lambda x)^T (y - \lambda x) + (z - \lambda y)^T (z - \lambda y) + x^T w$
subject to
$$\begin{aligned}
& w = Az + By + Cx \\
& e^T x = 1 \\
& e^T y = \lambda \\
& e^T z = \gamma \\
& \gamma - 2l\lambda + l^2 \geq 0 \\
& \gamma - 2u\lambda + u^2 \geq 0 \\
& (u + l)\lambda - \gamma - ul \geq 0 \\
& l \leq \lambda \leq u \\
& lx_j \leq y_j \leq ux_j\,, \quad \forall j = 1, \ldots, n \\
& rx_j \leq z_j \leq sx_j\,, \quad \forall j = 1, \ldots, n \\
& ly_j - l^2 x_j \leq z_j - ly_j \leq uy_j - lux_j\,, \quad \forall j = 1, \ldots, n \\
& lux_j - ly_j \leq uy_j - z_j \leq u^2 x_j - uy_j\,, \quad \forall j = 1, \ldots, n \\
& w \geq 0, x \geq 0, z \geq 0, \gamma \geq 0,
\end{aligned}$$

where we explicitly retain $z \geq 0$ and $\gamma \geq 0$ above for clarity and convenience in implementation, although these restrictions are implied by the other constraints in Problem P5. The enumerative method searches for a global minimum to Problem P5. As before, two types of branching strategies are used, namely, partitioning on a complementary pair of variables and by partitioning the interval $[l, u]$ for the variable $\lambda$. For any given node $k$, the restrictions imposed on the branches in the path from this node to the root define the corresponding node subproblem. Let us assume that these constraints are effectively given by

$$\begin{gathered}
\bar{l} \leq \lambda \leq \bar{u} \\
z_i = y_i = x_i = 0, \quad \forall i \in J \\
w_i = 0, \quad \forall i \in I,
\end{gathered}$$

where $l \leq \bar{l} < \bar{u} \leq u$, $J \subseteq \{1, \ldots, n\}$, $I \subseteq \{1, \ldots, n\}$, and $J \cap I = \emptyset$. Note that corresponding to these bounds on $\lambda$, we define as before $\bar{r} = \min\limits_{\lambda \in [\bar{l}, \bar{u}]} \lambda^2 < \bar{s} = \max\limits_{\lambda \in [\bar{l}, \bar{u}]} \lambda^2$. Furthermore, consider the sets

$$K = I \cup J,\ \bar{I} = \{1, \ldots, n\} \backslash I, \bar{J} = \{1, \ldots, n\} \backslash J, \quad \text{and} \quad \bar{K} = \{1, \ldots, n\} \backslash K.$$

Then the subproblem at node $k$ is given as follows:

$$
\begin{array}{lll}
\text{P5}(k) & \text{Minimize} & f(x,y,w,z,\lambda) = (y_{\bar J} - \lambda x_{\bar J})^T (y_{\bar J} - \lambda x_{\bar J}) + (z_{\bar J} - \lambda y_{\bar J})^T (z_{\bar J} - \lambda y_{\bar J}) + x_{\bar K}^T w_{\bar K} \\
 & \text{subject to} & w = Az + By + Cx \\
 & & e^T x_{\bar J} = 1 \\
 & & e^T y_{\bar J} = \lambda \\
 & & e^T z_{\bar J} = \gamma \\
 & & \gamma - 2\bar l \lambda + \bar l^2 \geq 0 \\
 & & \gamma - 2\bar u \lambda + \bar u^2 \geq 0 \\
 & & (\bar u + \bar l)\lambda - \gamma - \bar u \bar l \geq 0 \\
 & & \bar l \leq \lambda \leq \bar u \\
 & & \bar l x_j \leq y_j \leq \bar u x_j\,, \quad \forall j \in \bar J \\
 & & \bar r x_j \leq z_j \leq \bar s x_j\,, \quad \forall j \in \bar J \\
 & & \bar l y_j - \bar l^2 x_j \leq z_j - \bar l y_j \leq \bar u y_j - \bar l \bar u x_j\,, \quad \forall j \in \bar J \\
 & & \bar l \bar u x_j - \bar l y_j \leq \bar u y_j - z_j \leq \bar u^2 x_j - \bar u y_j\,, \quad \forall j \in \bar J \\
 & & w_{\bar I} \geq 0, x_{\bar J} \geq 0, z_{\bar J} \geq 0, \gamma \geq 0 \\
 & & z_j = y_j = x_j = 0, \quad \forall j \in J \\
 & & w_i = 0\,, \quad \forall i \in I.
\end{array}
$$

At this node $k$ of the tree, the algorithm searches for a stationary point to the corresponding program P5($k$). If the objective function value at this stationary point is zero, then a solution to the QEiCP is at hand and the algorithm terminates. Otherwise, two new nodes are created from node $k$ and the process is repeated. The algorithm also includes heuristic rules for choosing an open node from an associated list and for deciding which of the two branching strategies should be used at the selected node $k$ whenever a stationary point having a positive objective function value has been found for Problem P5($k$). The formal steps of the algorithm are identical to those of the procedure presented for EiCP in Section 2, except that the vector $(x, y, w, \lambda)$ is now replaced by $(x, y, w, z, \lambda)$, the node subproblem P1($k$) is replaced by P5($k$), and the computation $\theta_2$ at Step 2 is given by

$$\theta_2 = \max\{|\bar y_i - \bar\lambda \bar x_i|, |\bar z_i - \bar\lambda \bar y_i| \colon i \in \bar J\}.$$

The convergence of the algorithm is also guaranteed following the same argument used in Theorem 4.1 of [1]. To ensure finiteness, whenever $\bar u - \bar l \leq \epsilon$ at any node for some tolerance $\epsilon > 0$, we replace the corresponding lower and upper bounds on $\lambda$ by the common value $\frac{\bar u + \bar l}{2}$ at this node.

## 7 Computational Experience

In this section, we report some computational experience with the enumerative algorithm discussed in Sections 2 and 6 in order to illustrate its efficiency in computing eigenvalues for EiCP and QEiCP. All the tests have been performed on a Pentium IV (Intel) with Hyperthreading, 3.0 GHz CPU, 2GB RAM computer, using the operating system Linux. The algorithm was implemented in the General Algebraic Modeling System (GAMS) language (Rev 118 Linux/Intel) [26] and the NLP solver MINOS (Version 9.1) [27] was used to solve the subproblems at each node of the enumerative tree.

For the EiCP, we considered three sets of test problems, where $B$ was taken as the identity matrix. In the first set of test problems, the matrices A were taken from [9] and are given by

$$A = -\begin{bmatrix} 8 & -1 & 4 \\ 3 & 4 & 1/2 \\ 2 & -1/2 & 6 \end{bmatrix} \qquad \text{and} \qquad A = -\begin{bmatrix} 100 & 106 & -18 & -81 \\ 92 & 158 & -24 & -101 \\ 2 & 44 & 37 & -7 \\ 21 & 38 & 0 & 2 \end{bmatrix}.$$

These are denoted by AdlySeeger($n$), where $n$ is the order of the matrices ($n = 3$ and $4$, respectively). For the second set of test problems [28], the matrix $A$ is given by

$$A = -\begin{bmatrix} s^2 & s^3 & s^4 & s^5 & \dots \\ -s^3 & s^4 & s^5 & s^6 & \dots \\ -s^4 & s^5 & s^6 & s^7 & \dots \\ -s^5 & s^6 & s^7 & s^8 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

These test problems are denoted by Seeger($n$), where $n$ is the order of the matrices $A$ and $B$, and where we used $s = 3/2$. In the remaining set of test problems, the matrix $A$ was randomly generated with elements uniformly distributed in the intervals $[0, 1]$, $[-1, -1]$, $[-10, 10]$, and $[-100, 100]$. These problems are denoted by RAND$(k, m, n)$, where $k$ and $m$ are the end-points of the chosen interval for generating the matrix elements, and $n$ represents the order of the matrices $A$ and $B$ (we considered $n = 5, 10, 20, 30, 40, 50$, and $100$).

Table 1 displays the results obtained for solving the foregoing three sets of EiCP test problems via the enumerative algorithm. Our experience with these and similar test problems has indicated that the value of the tolerance $\epsilon$ used to terminate the algorithm should not be the same for $\theta_1$ and $\theta_2$ (see Step 2 of the algorithm) but instead two different tolerances $\epsilon_1$ and $\epsilon_2$ should be used for these quantities $\theta_i$, $i = 1, 2$, with $\epsilon_1$ being smaller than $\epsilon_2$. In Table 1 we report the results with $\epsilon_1 = 10^{-5}$ and $\epsilon_2 = 10^{-4}$. Our experience has shown that $\epsilon_1$ could be smaller than $10^{-5}$ (the algorithm works well in general even with $\epsilon_1 = 10^{-8}$) but $\epsilon_2$ should be chosen to be at least $10^{-4}$. For instance, for problem Seeger(20), $\epsilon_2$ has been set equal to $10^{-3}$ for the algorithm to terminate with a solution, because an indication of no solution occurred when $\epsilon_2 = 10^{-4}$ (this is impossible as the EiCP always has a solution $\lambda \in [l, u]$).

The lower bound $l$ is given by the optimal value of the program P2 (20)-(24). On the other hand, the upper bound $u$ is computed by $u = \min\{u_1, u_2\}$, where $u_1$ and $u_2$ are the values given by Theorem 3 and Theorem 4, respectively. The notation $\lambda$ stands for the eigenvalue computed by the algorithm. Finally, CPU, ITpivot, and Nodes are the total CPU time in seconds, the number of pivotal operations required, and the number of nodes enumerated by the algorithm, respectively (node = 0 means that no branching was performed).

The numerical results clearly indicate that the enumerative algorithm is efficient for solving all the linear EiCPs. The method enumerated only a few nodes to terminate and in many cases no branching was required. Another interesting conclusion from our experiments is that, in general, a complementary solution ($x_i w_i = 0$ for all $i = 1, \dots, n$) was typically found at the root node itself, and all the subsequent branchings were concerned with bracketing the interval for $\lambda$ until $y \approx \lambda x$ and a solution $(\lambda, x)$ was at hand. Furthermore, a local robust method such as the semi-smooth Newton's algorithm discussed in [9] could be incorporated in a hybrid method that uses the enumerative method first and switches to the Newton's method when $\theta_1$ is quite small (complementary solution is at hand) and $\theta_2$ is still larger than the desired tolerance. In such a hybrid procedure, the enumerative method should be considered as a safeguard. The analysis and implementation of such a hybrid method is recommended for future research.

In order to better assess the performance of the enumerative algorithm, we also solved all the test problems using the well-known and robust code BARON [11]. As for the enumerative method, BARON solves the EiCP by determining a global minimum of Problem P1(1) in Section 2 with $K = J = I = \emptyset$. The results of the performance of BARON (with default parameters settings) are displayed in Table 2. In this table, we use the notation $*$ to indicate when BARON was not able to find a solution to the EiCP. The reported numerical results indicate that BARON can solve many instances without branching, but it requires more effort than the enumerative algorithm whenever branching is required, which underscores the relative efficiency and efficacy of the proposed enumerative algorithm for solving the linear EiCP.

Table 3 reports the computational experience for solving QEiCPs using the enumerative algorithm. The first test problem, denoted by AdlySeegerQ(3), has been taken from [9]. In all the remaining test problems, the matrix $A$ was set equal to the identity matrix. Furthermore the matrices $B$ and $-C$ were randomly generated with their elements uniformly distributed in the intervals $[0, 1]$, $[0, 10]$, and $[0, 100]$. As for the EiCP, these test problems are denoted by RAND$(k, m, n)$, where $k$ and $m$ are the end-points of the chosen interval for generating the elements of the matrices $B$ and $-C$, and $n$ represents the order of the matrices $A$,

**Table 1** Performance of the enumerative algorithm for solving linear EiCPs.

| Problem | $l$ | $u$ | $\lambda$ | CPU | ITpivot | Nodes |
|---|---|---|---|---|---|---|
| AdlySeeger(3) | $-13.000$ | $1.718$ | $-4.134$ | 0.00 | 19 | 0 |
| AdlySeeger(4) | $-346.000$ | $224.157$ | $-29.134$ | 0.00 | 36 | 0 |
| Seeger(5) | $-150.214$ | $30.461$ | $-5.063$ | 0.00 | 26 | 0 |
| Seeger(10) | $-9802.776$ | $309.799$ | $-4.503$ | 0.00 | 62 | 0 |
| Seeger(20) | $-3.31620E+7$ | $22442.108$ | $-4.500$ | 0.00 | 98 | 0 |
| Seeger(30) | $-1.1030E+11$ | $1488244.077$ | $-4.500$ | 0.00 | 5 | 0 |
| Seeger(40) | $-3.6679E+14$ | $9.524743E+7$ | $-4.500$ | 0.00 | 6 | 0 |
| Seeger(50) | $-1.2197E+18$ | $5.971405E+9$ | $0.000$ | 0.00 | 1 | 0 |
| RAND(0,1,5) | $1.724$ | $3.475$ | $2.781$ | 0.00 | 15 | 0 |
| RAND(0,1,10) | $2.779$ | $5.906$ | $4.816$ | 0.00 | 35 | 0 |
| RAND(0,1,20) | $6.601$ | $11.756$ | $9.850$ | 0.00 | 66 | 0 |
| RAND(0,1,30) | $12.751$ | $18.389$ | $15.270$ | 0.00 | 94 | 0 |
| RAND(0,1,40) | $16.360$ | $23.839$ | $20.241$ | 0.00 | 131 | 0 |
| RAND(0,1,50) | $18.418$ | $29.756$ | $25.077$ | 0.01 | 166 | 0 |
| RAND(0,1,100) | $42.678$ | $55.350$ | $49.755$ | 0.05 | 381 | 0 |
| RAND(-1,1,5) | $-0.884$ | $3.504$ | $1.123$ | 0.00 | 24 | 0 |
| RAND(-1,1,10) | $-2.914$ | $5.399$ | $-0.018$ | 0.00 | 41 | 0 |
| RAND(-1,1,20) | $-3.684$ | $12.661$ | $0.842$ | 0.00 | 105 | 0 |
| RAND(-1,1,30) | $-8.048$ | $17.528$ | $2.346$ | 0.02 | 1235 | 9 |
| RAND(-1,1,40) | $-6.240$ | $22.948$ | $2.861$ | 0.07 | 3107 | 17 |
| RAND(-1,1,50) | $-8.005$ | $29.120$ | $3.131$ | 0.16 | 3239 | 9 |
| RAND(-1,1,100) | $-13.533$ | $56.866$ | $4.010$ | 0.84 | 5500 | 6 |
| RAND(-10,10,5) | $-19.796$ | $27.930$ | $-9.922$ | 0.00 | 29 | 2 |
| RAND(-10,10,10) | $-47.389$ | $67.795$ | $17.272$ | 0.00 | 113 | 0 |
| RAND(-10,10,20) | $-33.749$ | $121.968$ | $21.191$ | 0.03 | 554 | 6 |
| RAND(-10,10,30) | $-70.618$ | $179.965$ | $25.332$ | 0.03 | 1562 | 6 |
| RAND(-10,10,40) | $-62.677$ | $232.994$ | $19.595$ | 0.09 | 3130 | 11 |
| RAND(-10,10,50) | $-98.166$ | $288.328$ | $20.457$ | 0.16 | 3160 | 6 |
| RAND(-10,10,100) | $-180.103$ | $564.336$ | $39.242$ | 3.53 | 23928 | 16 |
| RAND(-100,100,5) | $-19.553$ | $327.351$ | $135.146$ | 0.00 | 23 | 0 |
| RAND(-100,100,10) | $-309.067$ | $611.982$ | $-40.854$ | 0.00 | 88 | 0 |
| RAND(-100,100,20) | $-485.177$ | $1192.792$ | $80.072$ | 0.02 | 378 | 0 |
| RAND(-100,100,30) | $-919.108$ | $1657.759$ | $180.221$ | 0.05 | 1812 | 5 |
| RAND(-100,100,40) | $-853.383$ | $2275.320$ | $234.283$ | 0.11 | 2293 | 3 |
| RAND(-100,100,50) | $-1345.659$ | $2905.844$ | $176.117$ | 0.31 | 4650 | 4 |
| RAND(-100,100,100) | $-1182.080$ | $5743.010$ | $526.975$ | 1.23 | 7246 | 2 |

$B$ and $C$. These choices of $A$, $B$, and $C$ imply that $(A, B, C)$ belongs to the class $\Gamma$, that is, the QEiCP is co-regular and co-hyperbolic and always has a solution. As before, the tolerances for termination were set to $\epsilon_1 = 10^{-5}$ and $\epsilon_2 = 10^{-4}$. The lower and upper bounds for the interval $[l, u]$ were computed according to the procedures described in Section 5. Note that $A$ is PD and $B^T$ is an S-matrix. Furthermore $u_1 = 0$ since $B$ is a copositive matrix. Finally $\lambda$, CPU, ITpivot, and Nodes have the same interpretation as before.

The numerical results displayed in Table 3 indicate that the enumerative algorithm was able to solve all the QEiCPs very efficiently (within 3 CPU seconds). In general, the algorithm required a small number of nodes to terminate and a complementary solution was in many cases found at the root node. For further improvements, we advocate the development of a similar hybrid method for solving the QEiCP as discussed above for the EiCP. As before, we also implemented BARON for solving the QEiCP by finding a global minimum of the nonlinear program P5 defined in Section 6. The numerical results for this experiment are displayed in Table 4 and indicate that BARON again required significantly greater effort than the enumerative algorithm, particulary when it was unable to solve the problem at the root node itself, and moreover, it failed to terminate with a solution to the QEiCP for one test problem (this is denoted by $*$).

## 8 Conclusions

In this paper, we have studied linear (EiCP) and quadratic (QEiCP) eigenvalue problems. Some properties concerned with the existence of a solution and with solutions to certain judiciously formulated nonlinear programming (NLP) problems were presented. An enumerative algorithm was proposed, which finds a

**Table 2** Performance of BARON for solving linear EiCPs.

| Problem | $l$ | $u$ | $\lambda$ | CPU | Nodes |
|---|---|---|---|---|---|
| AdlySeeger(3) | $-13.000$ | $1.718$ | $-4.134$ | 0.00 | 0 |
| AdlySeeger(4) | $-346.000$ | $224.157$ | $-32.864$ | 0.00 | 0 |
| Seeger(5) | $-150.214$ | $30.461$ | $-5.063$ | 0.00 | 0 |
| Seeger(10) | $-9802.776$ | $309.799$ | $-4.536$ | 0.00 | 0 |
| Seeger(20) | $-3.31620E+7$ | $22442.108$ | $-1.99032E+7$ | 11.01 | 9 |
| Seeger(30) | $-1.1030E+11$ | $1488244.077$ | $*$ | $*$ | $*$ |
| Seeger(40) | $-3.6679E+14$ | $9.524743E+7$ | $*$ | $*$ | $*$ |
| Seeger(50) | $-1.2197E+18$ | $5.971405E+9$ | $*$ | $*$ | $*$ |
| RAND(0,1,5) | $1.724$ | $3.475$ | $2.781$ | 0.01 | 0 |
| RAND(0,1,10) | $2.779$ | $5.906$ | $4.816$ | 0.00 | 0 |
| RAND(0,1,20) | $6.601$ | $11.756$ | $9.850$ | 0.01 | 0 |
| RAND(0,1,30) | $12.751$ | $18.389$ | $15.270$ | 0.03 | 0 |
| RAND(0,1,40) | $16.360$ | $23.839$ | $20.241$ | 0.06 | 0 |
| RAND(0,1,50) | $18.418$ | $29.756$ | $25.077$ | 0.11 | 0 |
| RAND(0,1,100) | $42.678$ | $55.350$ | $49.755$ | 0.79 | 0 |
| RAND(-1,1,5) | $-0.884$ | $3.504$ | $1.123$ | 0.00 | 0 |
| RAND(-1,1,10) | $-2.914$ | $5.399$ | $0.203$ | 0.00 | 0 |
| RAND(-1,1,20) | $-3.684$ | $12.661$ | $0.826$ | 0.01 | 0 |
| RAND(-1,1,30) | $-8.048$ | $17.528$ | $2.346$ | 288.64 | 307 |
| RAND(-1,1,40) | $-6.240$ | $22.948$ | $2.861$ | 323.09 | 100 |
| RAND(-1,1,50) | $-8.005$ | $29.120$ | $2.031$ | 164.94 | 145 |
| RAND(-1,1,100) | $-13.533$ | $56.866$ | $3.785$ | 1.58 | 0 |
| RAND(-10,10,5) | $-19.796$ | $27.930$ | $8.226$ | 0.00 | 0 |
| RAND(-10,10,10) | $-47.389$ | $67.795$ | $17.272$ | 0.00 | 0 |
| RAND(-10,10,20) | $-33.749$ | $121.968$ | $21.191$ | 21.45 | 46 |
| RAND(-10,10,30) | $-70.618$ | $179.965$ | $25.332$ | 215.43 | 136 |
| RAND(-10,10,40) | $-62.677$ | $232.994$ | $19.178$ | 80.36 | 28 |
| RAND(-10,10,50) | $-98.166$ | $288.328$ | $20.457$ | 143.89 | 19 |
| RAND(-10,10,100) | $-180.103$ | $564.336$ | $*$ | $*$ | $*$ |
| RAND(-100,100,5) | $-19.553$ | $327.351$ | $135.146$ | 0.00 | 0 |
| RAND(-100,100,10) | $-309.067$ | $611.982$ | $-44.055$ | 0.78 | 10 |
| RAND(-100,100,20) | $-485.177$ | $1192.792$ | $80.072$ | 0.32 | 1 |
| RAND(-100,100,30) | $-919.108$ | $1657.759$ | $180.221$ | 0.96 | 1 |
| RAND(-100,100,40) | $-853.383$ | $2275.320$ | $234.283$ | 7.93 | 1 |
| RAND(-100,100,50) | $-1345.659$ | $2905.844$ | $165.593$ | 15.74 | 1 |
| RAND(-100,100,100) | $-1182.080$ | $5743.010$ | $*$ | $*$ | $*$ |

**Table 3** Performance of the enumerative algorithm for solving QEiCPs.

| Problem | $l$ | $u$ | $\lambda$ | CPU | ITpivot | Nodes |
|---|---|---|---|---|---|---|
| AdlySeegerQ(3) | $-10.875$ | $5.469$ | $0.266$ | 0.00 | 10 | 0 |
| RAND(0,1,5) | $-4.944$ | $2.669$ | $0.708$ | 0.00 | 32 | 0 |
| RAND(0,1,10) | $-9.345$ | $4.903$ | $-5.575$ | 0.00 | 261 | 4 |
| RAND(0,1,20) | $-19.596$ | $10.042$ | $1.114$ | 0.01 | 229 | 0 |
| RAND(0,1,30) | $-29.585$ | $15.037$ | $1.055$ | 0.07 | 959 | 3 |
| RAND(0,1,40) | $-39.555$ | $20.022$ | $-21.182$ | 0.19 | 2078 | 4 |
| RAND(0,1,50) | $-49.273$ | $24.886$ | $1.127$ | 0.43 | 3997 | 8 |
| RAND(0,10,5) | $-42.789$ | $21.607$ | $-21.760$ | 0.00 | 138 | 4 |
| RAND(0,10,10) | $-95.230$ | $47.858$ | $0.968$ | 0.01 | 211 | 3 |
| RAND(0,10,20) | $-188.383$ | $94.447$ | $1.193$ | 0.02 | 484 | 2 |
| RAND(0,10,30) | $-289.527$ | $145.014$ | $-149.446$ | 0.13 | 2900 | 8 |
| RAND(0,10,40) | $-389.041$ | $194.772$ | $-197.925$ | 0.23 | 3136 | 6 |
| RAND(0,10,50) | $-489.202$ | $244.850$ | $-252.888$ | 0.65 | 7605 | 10 |
| RAND(0,100,5) | $-439.463$ | $219.978$ | $1.112$ | 0.00 | 46 | 0 |
| RAND(0,100,10) | $-930.600$ | $465.548$ | $0.887$ | 0.00 | 99 | 0 |
| RAND(0,100,20) | $-1863.996$ | $932.255$ | $1.953$ | 0.40 | 5620 | 38 |
| RAND(0,100,30) | $-2906.336$ | $1453.417$ | $-1494.046$ | 0.11 | 2164 | 4 |
| RAND(0,100,40) | $-3893.380$ | $1946.940$ | $1.077$ | 2.26 | 49894 | 62 |
| RAND(0,100,50) | $-4896.833$ | $2448.667$ | $-2488.298$ | 0.62 | 5186 | 4 |

**Table 4** Performance of BARON for solving QEiCPs.

| Problem | $l$ | $u$ | $\lambda$ | CPU | Nodes |
|---|---|---|---|---|---|
| SeegerAdlyQ(3) | $-10.875$ | 5.469 | 0.000 | 0.00 | 0 |
| RAND(0,1,5) | $-4.944$ | 2.669 | 0.842 | 0.00 | 0 |
| RAND(0,1,10) | $-9.345$ | 4.903 | $-5.575$ | 0.01 | 0 |
| RAND(0,1,20) | $-19.596$ | 10.042 | $-10.783$ | 0.98 | 1 |
| RAND(0,1,30) | $-29.585$ | 15.037 | $-16.177$ | 3.55 | 1 |
| RAND(0,1,40) | $-39.555$ | 20.022 | $-21.182$ | 8.56 | 1 |
| RAND(0,1,50) | $-49.273$ | 24.886 | $-26.020$ | 18.82 | 1 |
| RAND(0,10,5) | $-42.789$ | 21.607 | $-21.760$ | 0.33 | 6 |
| RAND(0,10,10) | $-95.230$ | 47.858 | $-56.093$ | 1.85 | 6 |
| RAND(0,10,20) | $-188.383$ | 94.447 | $-96.187$ | 16.46 | 7 |
| RAND(0,10,30) | $-289.527$ | 145.014 | $-149.446$ | 55.77 | 8 |
| RAND(0,10,40) | $-389.041$ | 194.772 | $-197.925$ | 144.50 | 10 |
| RAND(0,10,50) | $-489.202$ | 244.850 | $*$ | $*$ | $*$ |
| RAND(0,100,5) | $-439.463$ | 219.978 | 1.112 | 0.02 | 1 |
| RAND(0,100,10) | $-930.600$ | 465.548 | $-510.000$ | 1.91 | 10 |
| RAND(0,100,20) | $-1863.996$ | 932.255 | $-952.937$ | 20.63 | 8 |
| RAND(0,100,30) | $-2906.336$ | 1453.417 | $-1494.046$ | 64.01 | 8 |
| RAND(0,100,40) | $-3893.380$ | 1946.940 | 1.565 | 1647.56 | 91 |
| RAND(0,100,50) | $-4896.833$ | 2448.667 | $-2488.298$ | 376.09 | 8 |

solution to these problems by computing a global minimum of the formulated NLP. Computational results were presented to demonstrate the efficiency and efficacy of the enumerative algorithm for solving linear and quadratic EiCPs. The proposed algorithm was always able to find a solution for our set of test problems by enumerating only a few nodes, and was seen to be significantly more efficient than the well-known commercial software BARON for the same set of instances. The possible incorporation of a local search method, such as a semi-smooth Newton method [9, 19], in the final stages of the enumerative method could improve its efficiency and is worthy of further investigation. Alternative techniques for computing bounds on the eigenvalues as stated in Remark 1 are also worth exploring. The use of the enumerative method in a parametric algorithm for finding all the eigenvalues is another interesting topic of our ongoing research.

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