A 3D Magnetic Induction Tomography Numerical Forward Problem Solver Based on Finite Differences

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Abstract — This paper describes a 3D numerical model simulator developed to solve the forward problem of Magnetic Induction Tomography (MIT) for biological tissues, based on finite-differences. It is aimed to simulate experimental setups in harmonic, linear quasistatic conditions. A small presentation of the MIT paradigm and the theoretical electromagnetic background under the implementation of the simulator are presented in detail. The numerical method used and some testing against analytic solutions are explained. Furthermore, an implementation example in a real experimental situation is included, where 30 sensing coil acquisitions are simulated. Finally, some conclusions about future developments are presented.

I. INTRODUCTION

Magnetic Induction Tomography (MIT) is an imaging technique for assessing passive electrical properties of a body, based on the measurement of a small magnetic field, $\mathbf{AB}$, created by eddy currents induced in it by a magnetic field source $\mathbf{B}$ [1].

A typical MIT system is composed of an excitation coil and several receiver coils. Usually, alternating signals with tens of kHz to some MHz are used in the excitation coil, and the distance between receivers and emitter coils are in the order of centimeters. For these frequencies and distances over biological tissues, the diffusion nature of the electromagnetic equations prevails and the complex conductivity turns out to be real, resulting in a $\mathbf{AB}$ mostly influenced by conductivity and permeability maps. These properties define the specificity of this problem: permeability is almost constant and permeability maps. These properties define the equations of electromagnetic fields in these tissues.

In literature, MIT reconstruction algorithms uses a matrix sensitivity ($\mathbf{S}$), with elements equal to the relative electromotive voltage change in the n sensing coil due to the change of conductivity in a specific place k. Equation (1) presents the matrix $\mathbf{S}$, where $\Delta V_n$ is the n coil voltage change due to a $\Delta \sigma_k$ modification of conductivity in point k and $V$ is the received voltage in that coil.

$$\mathbf{S}_{n,k} = \frac{(\Delta V_n/V)}{\Delta \sigma_k} \quad (1)$$

Reconstruction techniques usually involve the inversion of this matrix, assuming a linearized relation.

The estimation of $\mathbf{S}$ can be done by three different approaches: Numerical based forward problem solvers, experimental testing and analytic solvers. Another approach is based on the use of a forward problem solver to find each value used to invert the solution. The numerical methods used in this step need to exhibit high performance in order to be solved in a short time frame. Commercial solvers are not suited for integration in an inverse problem and are not optimized for this specific problem.

The presented 3D solver was developed and tested in order to be able to simulate experimental data and to be subsequently incorporated in an inverse problem solver.

II. THEORETICAL BACKGROUND

Consider a typical eddy current problem depicted in Fig. 1. Air region is delimited by $\Gamma$ and the interface boundary between Air and a Body is called $\Gamma_{12}$. In the Air region conductivity is null and there are source currents. The Body region has non zero finite conductivity.

![Fig. 1. Problem concept](https://example.com/figure1.png)

Given the MIT context, it was used a linear, quasi-static approximation with the harmonic description of Maxwell equations. The governing equations are given by:

$$\nabla \times \mathbf{E} = -io\mathbf{B} \quad (2)$$
$$\nabla \times \mathbf{H} = \mathbf{J}_{\text{imposed}} + \nabla \phi \quad (3)$$
$$\nabla \cdot \mathbf{E} = 0 \quad (4)$$
$$\nabla \cdot \mathbf{B} = 0 \quad (5)$$

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With the constitutive relations:

\[ B = \mu H \]
\[ J_{\text{total}} = \nabla \Phi \]

Using the \( A-V \) formulation in (3) and current continuity equation, the following four differential equations are obtained [3, 4]:

\[ \nabla \times \nabla \times A + i\omega \mu_0 A = \mu (J_{\text{imposed}} - \nabla \Phi) \]
\[ \nabla \cdot (\nabla V + i\omega A) = 0 \]

where

\[ E = -(\nabla V + i\omega A) \]

was used. Although (8) is, by itself, enough to ensure current continuity, since its divergence equals first left side term to zero, (9) is explicitly used to make the problem non-singular, since four variables are presented and only three equations are stated by (8). Finally, adopting the Coulomb gauge \( (\nabla \cdot A = 0) \), (8) is more easily applied:

\[ \nabla^2 A - i\omega \mu_0 A = \mu (-J_{\text{imposed}} + \nabla \Phi) \]

To close the equation system, boundary and interface conditions must be adopted. Boundary conditions depend on the defined border type. Non-homogenous Dirichlet border will be used, forcing \( n \times A = 0 \); In \( \Gamma_{12} \), the continuity of \( B_n \), and the continuity of \( H_i \), are compulsory conditions. If \( A \) is a continuous function, then the first condition is automatically satisfied. Continuity of \( H_i \) could be fulfilled by forcing the normal component of \( j \) to vanish, that is,

\[ \nabla V_n = -i\omega A_n \]

Nevertheless, this equation needs not to be applied since (11) fulfills it automatically in the interface.

C. The 3D finite differences solver

The solver was developed using Matlab. It is separated in the three classic numerical solver modules: a preprocessor, a processor and a postprocessor. The first, allows defining the space geometry of the problem, sensing coils and excitation coils dimensions, defining a 3D map of conductivities to be simulated, the sensing coils number and other relevant parameters. The second module is responsible for the algorithm itself. The third makes small calculations in order to show relevant information about the solution in a graphical interface. In order to be able to solve the system of equations defined by (9) and (11), 3D finite differences method was chosen. This method allows having short development time frames and, hence, quickly obtaining useful requirements for further developments. Geometrically, the method subdivides the space in a discrete mesh. Although non-orthogonal grids are feasible, for the sake of simplicity, an orthogonal equally spaced grid was used. All the first-order discrete differential operators were defined based on their continuous analogous operators. A cubic volume with a variable number of elements in each side, typically around 70, was used. The element neighbors of a cell are presented in the next figure:

Fig. 2. 3D Finite Differences cell with their 6 neighbors elements

The resulting equations for \( N \) element cells are defined below. The three system of equations defined in (13) corresponds to the discretization of (11) and (14) corresponds to (9):

\[ \left( \frac{-6}{\mu} - i\omega \sigma_0 h^2 \right) A_{0,n}^{x,y,z} + \sum_{i=1}^{6} \frac{1}{h} A_{i,n}^{x,y,z} = -h^2 J_{\text{imposed}}^{x,y,z} \]

\[ \sum_{i=1}^{3} \left[ -(c_{i+1} + c_{i+3}) v_{i+1}^{x,y,z} + c_{i+1} v_{i+1}^{x,y,z} + c_{i} v_{i}^{x,y,z} \right] = \]

Where \( x,y,z \) stands for each system of equations, \( n \) is the \( n \)th equation of a system of equations, and \( i \) corresponds to the \( i \)th element in the cell \( n \).

The resulting problems can be stated as four linear problems of the type \( \bar{A} = \bar{b} \), where \( x \) correspond to all the three \( A \) vector coordinates and to the scalar value \( V \), for all the discrete points that compose the grid. Each \( \bar{A} \) is a large sparse matrix and \( b \) corresponds to the non homogenous part of the equation subtracted by the Dirichlet known values of \( x \).

Since typical test bodies have small conductivities and have small dimensions, the modified field \( A \) generated by the eddy currents should not modify significantly the value of \( A \) in the boundary. There are authors who even ignore this small value in the calculation of the new \( A \) [5]. Hence, one applied non-homogeneous Dirichlet conditions in the \( \Gamma \) boundary, defined by the analytic value of \( A \), calculated by the integral explicit formula given by

\[ A(r) = \frac{1}{4\pi} \int_{r} \frac{F(r')r'\cdot dr'}{R} \]

Where \( r \) is the position of each point in which \( A \) is calculated, and \( R \) the distance between the source and each point [6]. Current sources were discretized, imposing a total current equal to the value calculated analytically using (15).
A coil turn model is presented in the next figure for a 73 points discretization:

![Discretization model of a turn of the source coil](image)

**Fig. 3.** Discretization model of a turn of the source coil

It was chosen to solve the system of equations iteratively, solving the equation (9) only inside the body allowing two different discretization levels, one for each equation and subdividing the complexity into two major blocks presented in the next figure flowchart:

![Iterative method flowchart](image)

**Fig. 4.** Iterative method flowchart

A relaxation factor is used to force a smoother convergence. This is done by imposing a relatively smaller array of values of $\vec{A}\vec{V}$ than the one found solving (9). Each $AX=b$ equation was solved using the Quasi-minimal residual method (QMR). Finally, given the solution of the map $A$ in all the volume, the electromotive force generated in the sensing coils is calculated using the next equation, where the integral is taken along each sensing coil.

$$V_{emf} = \int_\Omega \omega \vec{A} \cdot d\vec{l}$$ (16)

### III. RESULTS

The simulated system is presented in the next figure:

![Simulated system setup](image)

**Fig. 5.** Simulated system setup

It is shown a source coil in bold, five sensing coils and a body located anywhere at the middle, having a specific conductivity map. The diameter of the circle where the coils lie tangentially is 10 cm; the source coil has 18 turns with 2.8 cm radius; the sensing coils have 6 turns and 1.5 cm radius. The source coil is being excited by a 18 mA sinusoidal current working at 800 kHz. In the next simulations, 73 elements in each volume side were used.

#### A. Simulation tests without body

To test equation (11), a simple 3D system without any conductive region applied was compared with the corresponding analytic solution, for the same geometry defined in Fig. 5. The absolute values of $A$ are compared in a parallel to x line that passes through the middle of the volume and presented in Fig. 6.

![Comparison chart between simulated and analytical A vector](image)

**Fig. 6.** Comparison chart between simulated and analytical $A$ vector in a parallel to x line that passes through the center of the volume

In this case, the solution is the direct result of (11) and it is obtained without iterations since equation (9) is never used.

#### B. Simulation tests with a body

Tests with a body of 3 cm of depth and 1.5 cm radius placed axisymmetrically could be compared with analytical calculations in order to test the solver. That could be done using (12) applied to the source coil and the fact that, at each point, the current density that passes in a given point at a distance $r$ from the center of symmetry is given by:

$$J = \frac{\omega}{2\pi} \phi \sigma \vec{A} \cdot d\vec{l}$$ (17)

This formulation is a good approximation when the skin depth ($\delta$) is much bigger than the depth of the body. In this case, $\delta =0.56$, that is 20 times bigger than the object depth.
Graphical representation of analytical and modeled absolute values of generated density currents that appeared in the body are shown in the next figure:

![Graphical representation of analytical and modeled absolute values of generated density currents](image)

**Fig. 7. Generated density currents of an axisymmetric body**

In order to be able to see the effect of eddy currents generated in non-axisymmetrical objects, a body with higher conductivity was used ($\sigma = 800 \text{ S/m}$) instead of the biologic typical values. In the left figure a qualitative image of the $B+\Delta B$ line fields for a slice of the simulated volume situated at $y = 10\text{cm}$ are presented. It can be seen the effect of new current inside the body. In the right figure, the same line fields without body are shown.

![Magnetic field lines with (left) and without body (right)](image)

**Fig. 8. Magnetic field lines with (left) and without body (right)**

**C. Sensitivity values**

In a MIT experimental acquisition, the measurement of the $S$ value for each object is most relevant, since it gives measurement of the total change imposed by the body presence. In this section, sensitivity values resulting from the simulation with a body with the same dimensions as before is introduced. It was made using 30 sensing coils uniformly distributed along a circular line as shown in Fig. 5, and a body located axisymmetrically, with $z = 10\text{cm}$.

![Resulting sensitivity for 30 sensing coils around an axisymmetrical body](image)

**Fig. 9. Resulting sensitivity for 30 sensing coils around an axisymmetrical body**

**III. DISCUSSION AND FURTHER WORK**

Analytical tests of the simulator have been presented, together with a simulation of a MIT experimental process with qualitative evaluation. To completely validate the system, real world experimental tests to the solver should be carried out.

Some concerns to the future development should be taken. Namely, bigger objects should be subject of other performance issues since matrices are not so sparse and, in some cases, could be ill conditioned. Performance optimization must also be taken into consideration.

In future work, new methods will be investigated deeply. Among others, Mimetic Finite difference methods and variational methods based on Edge elements should be tested. An important issue has to do with the self adaptive capacity of the grid, since this property could be very important in the reconstruction process.

**REFERENCES**


