On the Quadratic Eigenvalue Complementarity Problem over a General Convex Cone

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Abstract

The solution of the Conic Quadratic Eigenvalue Complementarity Problem (CQEiCP) is firstly investigated without assuming symmetry on the matrices defining the problem. A new sufficient condition for existence of solutions of CQEiCP is presented, extending to arbitrary pointed, closed and convex cones a condition known to hold when the cone is the nonnegative

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orthant. We also address the symmetric CQEiCP where all its defining matrices are symmetric. We show that, assuming that two of its defining matrices are positive definite, this symmetric CQEiCP reduces to the computation of a stationary point of an appropriate merit function on a convex set. Furthermore, we discuss the use of the so called Spectral Projected Gradient (SPG) algorithm for solving CQEiCP when the cone of interest is the second-order Cone (SOCQEiCP). A new algorithm is designed for the computation of the projections required by the SPG method to deal with SOCQEiCP. Numerical results are included to illustrate the efficiency of the SPG method and the new projection technique in practice.

1 Introduction

Given matrices $B, C \in \mathbb{R}^{n\times n}$, the *Eigenvalue Complementarity Problem* (to be denoted EiCP(B, C), see e.g. [29] and [30]), consists of finding $(\lambda, x, w) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$w = \lambda Bx - Cx, \tag{1.1}$$

$$w \geq 0, x \geq 0, \tag{1.2}$$

$$x^t w = 0, \tag{1.3}$$

$$e^t x = 1, \tag{1.4}$$

with $e = (1, 1, \ldots, 1)^t \in \mathbb{R}^n$. The last normalization constraint has been introduced, without loss of generality, in order to prevent the x component of a solution to vanish. The matrix B is usually assumed to be positive definite (PD). The problem has many applications in engineering (see [1], [27] and [30]), and can be seen as a generalization of the well-known Generalized Eigenvalue Problem, denoted GEiP (see e.g. [18]). Indeed, GEiP consists of solving (1.1) with $w = 0$, and a solution (λ, x) of GEiP is just an eigenvalue and eigenvector of the matrix $B^{-1}C$ in the usual sense, when B is nonsingular. If a triplet (λ, x, w) solves EiCP, then the scalar λ is called a *complementary eigenvalue* and x is a *complementary eigenvector* associated to λ for the pair (B, C). The condition $x^t w = 0$ and the nonnegative requirements on x and w imply that either $x_i = 0$ or $w_i = 0$ for $1 \leq i \leq n$. These two variables are called complementary.

It is easy to prove that under strict copositivity of B, EiCP(B, C) always has a solution, because it can be reformulated as the Variational Inequality Problem VIP($\bar{F}, \Omega$) with feasible set

$\Omega = \{x \in \mathbb{R}^n \colon e^t x = 1, x \geq 0\}$ and operator $\bar{F} : \Omega \to \mathbb{R}^n$ given by

$$\bar{F}(x) = \frac{x^t C x}{x^t B x} B x - C x, \tag{1.5}$$

see [22]. Note that $\bar{F}$ is continuous in Ω as a consequence of the strict copositivity of B, and that Ω is convex and compact. It is well known that these two conditions ensure existence of solutions of VIP($\bar{F}, \Omega$), see [11]. In particular this result holds when B is PD (see [22]).

A number of techniques have been proposed for solving the EiCP and its extensions, see e.g. [2], [6], [14], [15], [20], [21], [22], [23], [26], [28], [29], [32] and [33].

Recently an extension of the EiCP has been introduced in [31], where some applications are highlighted. It has been named *Quadratic Eigenvalue Complementarity Problem* (QEiCP), and it differs from EiCP through the existence of an additional quadratic term on λ. Its formal definition follows.

Given $A, B, C \in \mathbb{R}^{n \times n}$, QEiCP($A, B, C$) consists of finding $(\lambda, x, w) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$w = \lambda^2 A x + \lambda B x + C x, \tag{1.6}$$

$$w \geq 0, x \geq 0, \tag{1.7}$$

$$x^t w = 0, \tag{1.8}$$

$$e^t x = 1, \tag{1.9}$$

where, as before, $e = (1, 1, \ldots, 1)^t \in \mathbb{R}^n$. As in the case of the EiCP, the normalization constraint (1.9) has been introduced, without loss of generality, for preventing the x component of a solution of the problem from vanishing. Note that QEiCP(A, B, C) reduces to EiCP($B, -C$) when $A = 0$. The λ component of a solution of QEiCP(A, B, C) is called a *quadratic complementary eigenvalue* for A, B, C, and the x component a *quadratic complementary eigenvector* for A, B, C associated to λ.

The case of the symmetric QEiCP, i.e., when A, B and C are symmetric matrices and $-C$ is the identity matrix, has been analyzed in [13], where each instance of QEiCP with $n \times n$ matrices is related to an instance of EiCP with $2n \times 2n$ matrices. A new approach for solving the nonsymmetric QEiCP by a similar reduction has been recently studied in [7].

In this paper, we study the Conic Quadratic Eigenvalue Complementarity Problem (CQEiCP). This problem has been introduced in [31] as an interesting extension of QEiCP. It is defined as follows.

Given $A, B, C \in \mathbb{R}^{n\times n}$, a closed, convex and pointed cone $\mathcal{K} \subset \mathbb{R}^n$ and a vector $a \in \text{int}(\mathcal{K}^*)$, CQEiCP$(A, B, C)$ consists of finding $(\lambda, x, w) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$w = \lambda^2 Ax + \lambda Bx + Cx, \tag{1.10}$$

$$x \in \mathcal{K}, \;\; w \in \mathcal{K}^*, \tag{1.11}$$

$$x^t w = 0, \tag{1.12}$$

$$a^t x = 1. \tag{1.13}$$

As before, the normalization constraint (1.13) prevents $x = 0$ from being a solution of the problem. When $A = 0$, i.e., when the first term in the right hand side of (1.10) is absent, CQEiCP becomes the so called Conic Eigenvalue Complementarity Problem. This problem is denoted by CEiCP(B, R) and is defined by the constraints (1.11), (1.12), (1.13) and

$$w = \lambda Bx - Rx,$$

which replaces (1.10). Hence CEiCP(B, R) = CQEiCP$(0, B, -R)$. It is known (see [32]), that CEiCP(B, R) has a solution whenever $\mathcal{K}$ is closed, convex and pointed and B is a PD matrix. CQEiCP may lack solutions even when the leading matrix A is PD. Indeed, if we consider CQEiCP$(I, 0, I)$ with an arbitrary cone $\mathcal{K}$, then premultiplying (1.10) by x and using (1.12), one gets $0 = (\lambda^2 + 1)\, \|x\|^2$, which has no solution $\lambda \in \mathbb{R}$ and $x \neq 0$. This difference between CEiCP and CQEiCP in terms of existence of solutions mirrors the elementary fact that linear equations in one real variable always have solutions, while quadratic equations may fail to have them.

Thus, the issue of conditions on (A, B, C) ensuring existence of solutions of CQEiCP(A, B, C) is a relevant one. In [31], the concepts of co-regularity and co-hyperbolicity of (A, B, C) were introduced, ensuring existence of solutions of CQEiCP(A, B, C). For the case of QEiCP (i.e., when $\mathcal{K} = \mathbb{R}^n_+$), it has been shown in [7] that existence of solutions of QEiCP is also guaranteed when the matrix A is strictly copositive and the matrix $-C$ is not an S_0 matrix. In order to establish this result, QEiCP is transformed into a $2n$-dimensional EiCP problem by using an auxiliary vector $y \in \mathbb{R}^n$ such that $y = \lambda x$.

In this paper we propose a new transformation of CQEiCP into CEiCP (for a general closed, convex and pointed cone $\mathcal{K}$), that differs from the one introduced in [7] by the introduction of a PD matrix E. Using this transformation, we will establish in Section 2 the existence of solutions of CQEiCP under hypotheses different from those demanded in [31].

In Section 3, we show that the solution of the symmetric CQEiCP (i.e., when the matrices A, B and C are symmetric), assuming that both A and $-C$ are PD matrices, reduces to the computation of a stationary point of a special fractional quadratic function on a particular convex subset of the cone $\mathcal{K} \times \mathcal{K}$. This result extends the property, proved in [32], that the symmetric CEiCP(B, C) with a positive definite matrix B can be solved by computing a stationary point of the so-called Rayleigh quotient on the set defined by the cone $\mathcal{K}$ and a normalization constraint. Furthermore, it also generalizes a similar result established in [13] for the symmetric QEiCP when A is PD matrix and $-C$ is the identity matrix.

In Section 4 we address the case in which the cone $\mathcal{K}$ is the second-order Cone, defined as follows:

$$\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \ldots \times \mathcal{K}_r, \tag{1.14}$$

where

$$\mathcal{K}_i = \{x^i \in \mathbb{R}^{n_i} : \|\bar{x}^i\| \leq x_0^i\} \subset \mathbb{R}^{n_i} \ \ (1 \leq i \leq r), \tag{1.15}$$

$$\sum_{i=1}^{r} n_i = n.$$

Then any $x \in \mathcal{K}$ takes the form

$$x = (x^1, \ldots, x^r) \in \mathbb{R}^n$$

where x^i, $i = 1 \ldots r$ is given by

$$x^i = (x_0^i, \bar{x}^i)$$

with $x_0^i \in \mathbb{R}$ and $\bar{x}^i \in \mathbb{R}^{n_i - 1}$. It is rather immediate that each $\mathcal{K}_i$ is pointed and *self-dual*, i.e., it satisfies $\mathcal{K}_i = \mathcal{K}_i^*$. As a consequence, the second-order Cone $\mathcal{K}$ is pointed and satisfies $\mathcal{K} = \mathcal{K}^*$ (see [3]).

Optimization problems whose feasible sets are second-order Cones are computationally tractable and appear in a large variety of applications, such as filter design, antenna array weight design, truss design, robust estimation and friction in robot grasp (see [3, 5, 24]). Recognizing the importance of such models, the second-order Eigenvalue Complementarity Problem (SOCEiCP) has been introduced in [1]. It is just the special case of CEiCP when $\mathcal{K}$ is the second-order Cone. Similarly, the second-order Cone Quadratic Eigenvalue Complementarity Problem (SOCQEiCP) is the particular case of CQEiCP when $\mathcal{K}$ is the second-order Cone. Algorithms for the nonsymmetric SOCEiCP have been discussed in [1] and [12]. In this paper we study the solution of the symmetric SOCEiCP and SOCQEiCP. As stated above, finding a solution of these problems reduces to the computation of a stationary point of a fractional quadratic program. As in [20], we propose the Spectral Projected Gradient (SPG) algorithm for computing such a stationary point. The efficiency of the algorithm depends on the computation of projections on the feasible (convex) set of the optimization problem. The normalization constraint

$$\sum_{i=1}^{r} x_0^i = 1 \tag{1.16}$$

is introduced, so that these projections can be computed efficiently by a new algorithm proposed in Section 4. Numerical results with the SPG algorithm, using this new technique for computing projections, are reported, showing the efficiency of this approach for solving the symmetric SOCEiCP and SOCQEiCP.

The paper is organized as follows. The sufficient condition for existence of solutions of CQEiCP is introduced in Section 2. The symmetric case is discussed in Section 3. The SPG algorithm for SOCEiCP and SOCQEiCP is described in Section 4. Numerical results with this algorithm are reported in Section 5 and some conclusions are presented in the last section of the paper.

2 Existence of solutions of CQEiCP

In this section we present a sufficient condition for the existence of solutions of CQEiCP(A, B, C). We start by recalling some basic facts about cones. A set $\mathcal{K} \subset \mathbb{R}^n$ is a *cone* when it is closed under multiplication by nonnegative scalars. We are concerned here with convex cones. It is easy to conclude that convex cones are precisely those subsets of $\mathbb{R}^n$ which are closed by linear combinations with nonnegative scalars. In this paper we consider exclusively closed convex cones, i.e. those convex cones which are closed in the standard topology in $\mathbb{R}^n$ (i.e. the topology induced by any norm). We recall that a cone $\mathcal{K}$ is *pointed* if it does not contain lines, or equivalently, if there exists no nonzero $x \in \mathcal{K}$ such that $-x \in \mathcal{K}$. We mention that any cone $\mathcal{K}$ can be written as $\mathcal{K} = \mathcal{K}' + L$ where "+" denotes the Minkowski sum, $\mathcal{K}'$ is pointed and L is a linear subspace (L is the *linearity* of $\mathcal{K}$, namely $L = \{x \in \mathcal{K} : -x \in \mathcal{K}\}$, and $\mathcal{K}'$ can be taken as $\mathcal{K}' = \mathcal{K} \cap L^{\perp}$; see, e.g., [17]). Given a cone $\mathcal{K}$, its *dual cone* (or *positive polar cone*) $\mathcal{K}^*$ is defined as $\mathcal{K}^* = \{x \in \mathbb{R}^n : x^t y \geq 0 \ \forall y \in \mathcal{K}\}$. It is elementary to check that $\mathcal{K}$ is pointed if and only if $\mathcal{K}^*$ has nonempty interior.

Now, we recall the sufficient conditions introduced in [31].

Definition 2.1. Consider a cone $\mathcal{K} \subset \mathbb{R}^n$.

i) A matrix $A \in \mathbb{R}^{n\times n}$ is $\mathcal{K}$-regular if $x^t A x \neq 0$ for all nonzero $x \in \mathcal{K}$.

ii) A triplet (A, B, C), with $A, B, C \in \mathbb{R}^{n\times n}$ is $\mathcal{K}$-hyperbolic if

$$(x^t B x)^2 \geq 4(x^t A x)(x^t C x) \tag{2.1}$$

for all nonzero $x \in \mathcal{K}$.

Theorem 2.2. *If $\mathcal{K}$ is a closed, convex and pointed cone, A is $\mathcal{K}$-regular and (A, B, C) is $\mathcal{K}$-hyperbolic, then CQEiCP(A, B, C) has solutions.*

Proof. See Theorem 3.3 in [31]. □

In this paper, we guarantee the existence of solutions of CQEiCP by a different approach based on the relationship between an arbitrary n-dimensional CQEiCP and two specific instances of CEiCP with matrices in $\mathbb{R}^{2n\times 2n}$.

Consider now CQEiCP(A, B, C) with $A, B, C \in \mathbb{R}^{n\times n}$, take any symmetric and positive definite matrix $E \in \mathbb{R}^{n\times n}$, and define the matrices $D, G, H \in \mathbb{R}^{2n\times 2n}$ as

$$D = \begin{pmatrix} A & 0 \\ 0 & E \end{pmatrix}, \tag{2.2}$$

$$G = \begin{pmatrix} -B & -C \\ E & 0 \end{pmatrix}, \tag{2.3}$$

$$H = \begin{pmatrix} B & -C \\ E & 0 \end{pmatrix}. \tag{2.4}$$

Given the cone $\mathcal{K} \subset \mathbb{R}^n$, we define the cone $\tilde{\mathcal{K}} \subset \mathbb{R}^{2n}$ as $\tilde{\mathcal{K}} = \mathcal{K} \times \mathcal{K}$. Furthermore, for a given $a \in \mathrm{int}(\mathcal{K}^*)$, we define $\tilde{a} \in \mathbb{R}^{2n}$ as $\tilde{a} = (a, a)$. Note that $\tilde{a}$ belongs to $\mathrm{int}(\tilde{\mathcal{K}})$. Assuming that the cone related to CQEiCP(A, B, C) is $\mathcal{K}$, and the vector in $\mathrm{int}(\mathcal{K}^*)$ appearing in (1.13) is a, we consider CEiCP(D, G) and CEiCP(D, H) with cone $\tilde{\mathcal{K}}$ and vector $\tilde{a}$.

Next we prove a relation between the solutions of CQEiCP(A, B, C) and those of CEiCP(D, G) and CEiCP(D, H). We emphasize that the following result holds without making any additional hypotheses on A, B, C. We also mention that the proof of Proposition 2.3(b) is quite different from the proof of its counterpart for the case of $\mathcal{K} = \mathbb{R}^n_+$, namely Proposition 1 in [7].

Proposition 2.3. a) Assume that (λ, x) solves CQEiCP(A, B, C) and consider D, G, H as in (2.2)–(2.4).

i) If $\lambda = 0$ then $(\lambda, z) = (0, z)$ solves both CEiCP(D, G) and CEiCP(D, H), where $z \in \mathbb{R}^{2n}$ is defined as $z = (0, x)$.

ii) If $\lambda > 0$ then (λ, z) solves CEiCP(D, G), where $z \in \mathbb{R}^{2n}$ is defined as $z = (1+\lambda)^{-1}(\lambda x, x)$.

iii) If $\lambda < 0$ then the pair $(-\lambda, z)$ solves CEiCP(D, H), where $z \in \mathbb{R}^{2n}$ is defined as $z = (1 - \lambda)^{-1}(-\lambda x, x)$.

b) Consider D, G, H as in (2.2)–(2.4).

i) If (λ, z) solves CEiCP(D, G) with $z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \neq 0$, then $\lambda > 0$ and $(\lambda, (1 + \lambda)x)$ solves CQEiCP(A, B, C)

ii) If (λ, z) solves CEiCP(D, H) with $z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \neq 0$, then $\lambda > 0$ and $(-\lambda, (1 + \lambda)x)$ solves CQEiCP(A, B, C).

Proof. a) For item (i), note that if $(0, x)$ solves CQEiCP(A, B, C) then it holds that $Cx \in \mathcal{K}^*, x \in \mathcal{K}$ and $x^t Cx = 0$. It is easy to check that these three conditions imply that the pair $(0, (0, x))$ solves both CEiCP(D, G) and CEiCP(D, H). We deal now with item (ii). Note that checking that a pair (λ, z) with $z = (u, v) \in \mathbb{R}^n \times \mathbb{R}^n$ solves CEiCP(D, G) is equivalent to verifying:

$$\lambda Au + Bu + Cv \in \mathcal{K}^*, \tag{2.5}$$

$$E(\lambda v - u) \in \mathcal{K}^*, \tag{2.6}$$

$$u \in \mathcal{K}, \quad v \in \mathcal{K}, \tag{2.7}$$

$$u^t(\lambda A u + B u + C v) + v^t E(\lambda v - u) = 0, \tag{2.8}$$

$$a^t(u + v) = 1. \tag{2.9}$$

On the other hand, since (λ, x) solves CQEiCP(A, B, C), we know that

$$\lambda^2 A x + \lambda B x + C x \in \mathcal{K}^*, \tag{2.10}$$

$$x \in \mathcal{K}, \tag{2.11}$$

$$x^t(\lambda^2 A x + \lambda B x + C x) = 0, \tag{2.12}$$

$$a^t x = 1. \tag{2.13}$$

If we take $u = \dfrac{\lambda}{1+\lambda} x, v = \dfrac{1}{1+\lambda} x$, then we have $\lambda v - u = 0$ and (2.6) holds trivially. The condition (2.5) follows from (2.10), and (2.7) follows from (2.11) and positivity of λ. The first term of the left hand side of (2.8) vanishes as a consequence of (2.12). Since $\lambda v = u$ then the equality (2.8) holds. Now $a^t(u + v) = (1 + \lambda)^{-1}(\lambda a^t x + a^t x) = a^t x = 1$ by (2.13). Hence, (2.9) holds. For item (iii), note that if (λ, x) solves CQEiCP(A, B, C) then $(-\lambda, x)$ solves CQEiCP$(A, -B, C)$. In such a case, as $-\lambda$ is positive, we can apply item (ii) to CQEiCP$(A, -B, C)$, replacing λ by $-\lambda$ and B by $-B$. This gives the result, taking into account the definitions of z and H.

b) Consider first item (i). We know that (2.5)–(2.9) hold with $(u, v) = (y, x)$, and we need to check that

$$(1 + \lambda)(\lambda^2 A x + \lambda B x + C x) \in \mathcal{K}^*, \tag{2.14}$$

$$(1 + \lambda)x \in \mathcal{K}, \tag{2.15}$$

$$(1 + \lambda)^2 \left[x^t(\lambda^2 A x + \lambda B x + C x)\right] = 0, \tag{2.16}$$

$$(1 + \lambda)a^t x = 1. \tag{2.17}$$

If $\lambda \geq 0$ then (2.15) follows immediately from (2.7). It is rather elementary to verify that if

$$y = \lambda x, \tag{2.18}$$

then (2.14) follows from (2.5), (2.16) follows from (2.12), and (2.17) follows from (2.13). Therefore $(\lambda, (1+\lambda)x)$ solves CQEiCP(A, B, C), provided $\lambda \geq 0$.

We prove next that (2.18) holds. We claim first that $x \neq 0$. Otherwise (2.6) gives $-Ey \in \mathcal{K}^*$. Since $y \in \mathcal{K}$ by (2.7), we get $-y^t Ey \geq 0$, which implies $y = 0$, because E is positive definite. Since $x = 0$, we have $a^t(x+y) = 0$, contradicting (2.9). Consider now (2.8). Note that each term in the left hand side is nonnegative, because x, y belong to $\mathcal{K}$, and $\lambda Ay + By + Cx, E(\lambda x - y)$ belong to $\mathcal{K}^*$, by (2.5)–(2.7). It follows that both terms vanish, and in particular the second one. Hence $0 = x^t E(\lambda x - y)$, i.e.

$$\lambda = \frac{x^t Ey}{x^t Ex}, \tag{2.19}$$

taking into account that $x \neq 0$, and hence $x^t Ex > 0$. It follows from (2.19) that $y \neq 0$, since both x and λ are known to be nonzero. On the other hand, since $E(\lambda x - y) \in \mathcal{K}^*$, $y \in \mathcal{K}$, we have

$$y^t Ey \leq \lambda x^t Ey. \tag{2.20}$$

Substituting (2.19) in (2.20), we obtain $(y^t Ey)(x^t Ex) \leq (y^t Ex)^2$. Define now $\langle \cdot, \cdot \rangle_E$, $\|\cdot\|_E$ as $\langle x, y \rangle_E = x^t Ey$, $\|x\|_E = (x^t Ex)^{1/2}$. By using the Cauchy-Schwartz inequality,

$$\|x\|_E \, \|y\|_E \leq |\langle x, y \rangle_E| \leq \|x\|_E \, \|y\|_E \,. \tag{2.21}$$

It follows from (2.21) that Cauchy-Schwartz inequality holds with equality. Therefore x and y are collinear, i.e. there exists $\sigma \in \mathbb{R}$ such that $y = \sigma x$. Replacing this equation in (2.19) and using the fact that $x \neq 0$, we conclude that $\lambda = \sigma$. Hence (2.18) holds.

Finally, positivity of λ follows also from (2.18). Since $(x, y) \in \tilde{\mathcal{K}}$, we get that $x \in \mathcal{K}$ and $\lambda x \in \mathcal{K}$, so that $\lambda < 0$ contradicts the pointedness of $\mathcal{K}$.

For item (ii), we apply the same argument as in item (i) to CEiCP(D, H). Since G and H differ just by the sign of B, we conclude that $(\lambda, (1+\lambda)x)$ solves CQEiCP$(A, -B, C)$. It now follows from the definition of CQEiCP(A, B, C) that $(-\lambda, (1+\lambda)x)$ solves it.

$\square$

We mention that a relationship between an n-dimensional instance of CQEiCP and $2n$-dimensional instances of CEiCP, similar to (2.2)-(2.4), but with the identity matrix substituting for E, has been considered in [7] for QEiCP. The consideration of a more general matrix E has some interesting computational consequences. Assume for instance that A, B, C are symmetric and that $-C$ is positive definite. Then taking $E = -C$ in (2.2)-(2.4) produces symmetric matrices D, G and H. In such a case CEiCP(D, G) and CEiCP(D, H) are considerably easier to solve, as discussed in the next section.

We comment also that our sufficient condition requires only item (b) of Proposition 2.3. However, item (a) has some interesting consequences, see Remarks 2.9 and 2.10 below.

Now we rephrase the result of Proposition 2.3 in terms of complementary eigenvalues.

Corollary 2.4. Consider CQEiCP(A, B, C) with $A, B, C \in \mathbb{R}^{n\times n}$ and the matrices $D, G, H \in \mathbb{R}^{2n\times 2n}$ as defined in (2.2)–(2.4). Then,

i) all quadratic complementary eigenvalues for (A, B, C) are complementary eigenvalues for either (D, G), or (D, H), or both,

ii) all nonzero complementary eigenvalues for (D, G) are positive, and are quadratic complementary eigenvalues for (A, B, C),

iii) all nonzero complementary eigenvalues for (D, H) are positive, and their additive inverses are quadratic complementary eigenvalues for (A, B, C).

Proof. Elementary from Proposition 2.3. □

Corollary 2.4 signals a clear path for obtaining a sufficient condition for existence of solutions of CQEiCP(A, B, C). We must first find a sufficient condition for solvability of CEiCP(D, G) or CEiCP(D, H) (which in principle depends only on the matrix in the leading term in (1.1), namely D in this case, and henceforth just on A, in terms of the data of the CQEiCP), and then impose conditions ensuring that either 0 is a quadratic complementary eigenvalue for (A, B, C), or that 0 is not a complementary eigenvalue of $(D, G), (D, H)$ (the second option depends only upon C and not on A, B; this fact was mentioned in the proof of Proposition 2.3 and can be easily checked).

We present next some classes of matrices needed for our sufficient conditions.

Definition 2.5. Consider a cone $\mathcal{K} \subset \mathbb{R}^n$.

i) A matrix $M \in \mathbb{R}^{n\times n}$ is said to be strictly $\mathcal{K}$-copositive if $x^t M x > 0$ for all $0 \neq x \in \mathcal{K}$.

ii) The class $R_0'(\mathcal{K}) \subset \mathbb{R}^{n\times n}$ consists of those matrices $M \in \mathbb{R}^{n\times n}$ such that $x^t M x = 0$ for at least a nonzero $x \in \mathcal{K}$ satisfying $Mx \in \mathcal{K}^*$.

iii) The class $S_0'(\mathcal{K}) \subset \mathbb{R}^{n\times n}$ consists of those matrices $M \in \mathbb{R}^{n\times n}$ such that there exists no nonzero $x \in \mathcal{K}$ such that $Mx \in \mathcal{K}^*$.

We comment that for $\mathcal{K} = \mathbb{R}^n_+$, the complements of classes $R_0'(\mathcal{K}), S_0'(\mathcal{K})$ are the well known classes S_0, R_0 respectively (see e.g. [9]).

Proposition 2.6. i) If $M \in \mathbb{R}^{n\times n}$ is strictly $\mathcal{K}$-copositive then CEiCP(M, C) has solutions for any $C \in \mathbb{R}^{n\times n}$.

ii) If $C \in R_0'(\mathcal{K})$ then 0 is a quadratic complementary eigenvalue for (A, B, C) for any $A, B \in \mathbb{R}^{n\times n}$.

iii) If $C \in S_0'(\mathcal{K})$ then 0 is not a complementary eigenvalue for either (D, G) or (D, H) with D, G, H as in (2.2)–(2.4).

Proof. Item (i) has been proved in [32], as mentioned in the introduction. Item (ii) is immediate from the definitions of CQEiCP and $R_0'(\mathcal{K})$. For item (iii), assume that 0 is a complementary eigenvalue for (D, G), with associated complementary eigenvector $0 \neq z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$. It follows immediately that $By + Cx \in \mathcal{K}^*$, $-Ey \in \mathcal{K}^*$, $x \in \mathcal{K}$, $y \in \mathcal{K}$. Hence, $-y^t E y \geq 0$, so that $y = 0$ because E is positive definite, and therefore $Cx \in \mathcal{K}^*$. As $z \neq 0$, it follows that $x \neq 0$, and we have a contradiction with the assumption that $C \in S_0'(\mathcal{K})$. The same argument can be used for the case of (D, H). □

Now, all the pieces are in place for stating and proving our existence result for CQEiCP.

Theorem 2.7. *Consider CQEiCP(A, B, C).*

i) $C \in R_0'(\mathcal{K})$ if and only if 0 is a quadratic complementary eigenvalue for QEiCP(A, B, C).

ii) If $C \in S_0'(\mathcal{K})$ and A is strictly $\mathcal{K}$-copositive, then there exist at least one positive and one negative quadratic complementary eigenvalue for (A, B, C).

Proof. Item (i) is a consequence of Proposition 2.6 (ii). For proving item (ii), we first note that strictly $\mathcal{K}$-copositivity of A implies strictly $\mathcal{K}$-copositivity of D. Hence both CEiCP(D, G) and CEiCP(D, H) have complementary eigenvalues by Proposition 2.6(i), which are nonzero by Proposition 2.6 (iii). Hence, they are positive by items (ii) and (iii) of Corollary 2.4. Therefore there exist at least one positive and one negative quadratic complementary eigenvalue for (A, B, C). $\square$

In the remainder of this section, we discuss the existence result given in Theorem 2.7. We start with a corollary, stating that the roles of A and C in item (ii) of Theorem 2.7 can be reversed.

Corollary 2.8. Consider CQEiCP(A, B, C) and assume that $A \in S_0'(\mathcal{K})$ and C is strictly $\mathcal{K}$-copositive. Then there exist at least one positive and one negative quadratic complementary eigenvalue for (A, B, C).

Proof. Apply Theorem 2.7(ii) to CQEiCP(C, B, A) and conclude that it has a solution (λ, x) with $\lambda > 0$, so that

$$w = \lambda^2 Cx + \lambda Bx + Ax \in \mathcal{K}^*, x \in \mathcal{K}, w^t x = 0. \tag{2.22}$$

Let $\mu = \lambda^{-1}$. Divide the first inequality in (2.22) by λ^2, and get from (2.22) $\bar{w} = \mu^2 Ax + \mu Bx + Cx \in \mathcal{K}^*, x \in \mathcal{K}, \bar{w}^t x = 0$, so that (μ, x) solves CQEiCP(A, B, C) and $\mu > 0$. Proceeding in the same fashion with CQEiCP$(C, -B, A)$, get a solution $(\bar{\lambda}, \bar{x})$ of this problem with $\bar{\lambda} > 0$, take $\bar{\mu} = \bar{\lambda}^{-1}$ and conclude that $(\bar{\mu}, \bar{x})$ solves CQEiCP$(A, -B, C)$. Hence $-\bar{\mu}$ is a negative quadratic complementary eigenvalue for (A, B, C). $\square$

We continue with two remarks related to the result in Theorem 2.7.

Remark 2.9. When we move from CQEiCP(A, B, C) to CEiCP(D, G), we can settle the issue of existence of solutions for the former except for one "undeterminated" case: when we only know that 0 is a complementary eigenvalue for (D, G). If EiCP(D, G) has no solutions then the same happens to CQEiCP(A, B, C) by Corollary 2.4(i); if CEiCP(D, G) has a solution (λ, x) with $\lambda \neq 0$ then λ is a quadratic complementary eigenvalue for (A, B, C) by Corollary 2.4(ii), but the fact that 0 is a complementary eigenvalue for (D, G) entails no conclusion at all about the existence of solutions of CQEiCP(A, B, C). The same considerations hold for CEiCP(D, H).

Remark 2.10. As another consequence of Corollary 2.4, if a method for finding all complementary eigenvalues for an arbitrary instance of CEiCP is available, applying it to CEiCP(D,G) and CEiCP(D,H) provides all quadratic complementary eigenvalues of CQEiCP(A,B,C). In fact, all complementary eigenvalues of these two CEiCP's are quadratic complementary eigenvalues for CQEiCP(A,B,C) (with the possible exception of 0, which can be checked separately) by virtue of Corollary 2.4 (ii)–(iii), and no quadratic complementary eigenvalue can be missed, as a consequence of Corollary 2.4(i).

Finally, we close the section with the comparison between the two sets of sufficient conditions for existence of solutions of CQEiCP(A,B,C) given in Theorems 2.2 and 2.7.

For the comparison between the assumptions of Theorem 2.2 and Theorem 2.7, we say that a triplet (A,B,C) satisfies (P) when either $C \in S_0'(\mathcal{K})$ and A is strictly $\mathcal{K}$-copositive, or $C \in R_0'(\mathcal{K})$, and that it satisfies (P') when A is $\mathcal{K}$-regular and (A,B,C) is $\mathcal{K}$-hyperbolic.

We mention that if both A and $-C$ are strictly $\mathcal{K}$-copositive, then (P') holds, because in such a case one has $x^tAx \geq 0$, $x^tCx \leq 0$ for all $x \in \mathcal{K}$, so that the right hand side in (2.1) is nonpositive, making this inequality valid.

On the other hand, it is easy to exhibit instances in which (P) holds but (P') does not. Indeed, consider any pointed cone $\mathcal{K}$ which is not a halfline (i.e., it contains at least two linearly independent vectors, say c, d), take $a \in \mathrm{int}(\mathcal{K}^*)$, find a vector $b \in \mathbb{R}^n$ such that $b^tc < 0$, $b^td > 0$, and define $C \in \mathbb{R}^{n\times n}$ as $C = ba^t$. We claim that if A is positive definite then the triplet $(A,0,C)$ satisfies (P) but not (P'). Observe that (2.1) fails with $x = d$, since

$$(d^tBd)^2 - 4(d^tAd)(d^tCd) = -4(d^tAd)(a^td)(b^td) < 0.$$

On the other hand, $(A,0,C)$ satisfies (P). Since A is positive definite, it is $\mathcal{K}$-copositive for all $\mathcal{K}$. For showing that $C \in S_0'(\mathcal{K})$, take any nonzero $x \in \mathcal{K}$. Hence $Cx = (a^tx)b$. If $Cx \in \mathcal{K}^*$, then $0 \leq (Cx)^tc = (a^tx)(b^tc) < 0$, as $a^tx > 0$ and $b^tc < 0$ by construction. Hence $Cx \notin \mathcal{K}^*$ and $C \notin S_0'(\mathcal{K})$.

There are also many instances of CQEiCP for which (P') holds but not (P). Take for instance an arbitrary $\mathcal{K}$, $A = C = I$ and $B = 2I$. Validity of (P') for any $\mathcal{K}$ is immediate, but (P) fails, because $I \notin R_0'(\mathcal{K}) \cup S_0'(\mathcal{K})$ for any $\mathcal{K}$. Hence, (P) and (P') are independent of each other for a generic cone $\mathcal{K}$.

Observe also that (P) depends only upon the matrices A and C, while (P') also involves the matrix B.

3 Symmetric CEiCP and CQEiCP

It has been proved in [32] that if B is $\mathcal{K}$-regular (as in Definition 2.1), then the set of solutions of CEiCP(B, C) coincides with the set of solutions of VIP$(\bar{F}, \Delta)$, with $\bar{F}$ as in (1.5) and $\Delta = \{x \in \mathcal{K} : a^t x = 1\}$. Now, it is well known that if $S \subset \mathbb{R}^n$ is a closed and convex set and $h : S \to \mathbb{R}$ is differentiable on an open set containing S, then a point $\bar{x} \in S$ satisfies the first order optimality condition for the problem of minimizing $h(x)$ subject to $x \in S$ if and only if

$$\nabla h(\bar{x})^t (x - \bar{x}) \geq 0 \ \ \forall x \in S, \tag{3.1}$$

which is the same as saying that $\bar{x}$ solves VIP$(\nabla h, S)$. Note that the condition (3.1) means that no direction starting at $\bar{x}$ and pointing to a point in S is a descent direction for h.

Hence, if there exists a function h such that the solutions of VIP$(\bar{F}, \Delta)$ coincide with those of VIP$(\nabla h, \Delta)$, then the solutions of CEiCP(B, C) are precisely the stationary points for the problem of minimizing h on Δ. This is the case when CEiCP(B, C) is *symmetric*, meaning that both B and C are symmetric matrices. Indeed, assume that B is $\mathcal{K}$-regular and consider $h : \mathcal{K} \to \mathbb{R}$ defined as

$$h(x) = -\frac{x^t C x}{x^t B x}. \tag{3.2}$$

We mention that the quotient in (3.2) is called the *Rayleigh quotient* for B, C. Note that $\mathcal{K}$-regularity of B implies that h is well defined (and indeed differentiable) in an open set containing Δ, and that its gradient is given by

$$\nabla h(x) = \frac{2}{x^t B x} \left[\frac{x^t C x}{x^t B x} B x - C x \right] = \frac{2}{x^t B x} \bar{F}(x). \tag{3.3}$$

Now, note that if B is $\mathcal{K}$-regular then either B is $\mathcal{K}$-copositive or $-B$ is $\mathcal{K}$-copositive. If B is $\mathcal{K}$-copositive, then it follows from (3.3) that $\nabla h(\bar{x})^t (x - \bar{x}) \geq 0$ if and only if $\bar{F}(\bar{x})^t (x - \bar{x}) \geq 0$, so that the solution sets of VIP$(\bar{F}, \Delta)$ and VIP$(\nabla h, \Delta)$ coincide. If $-B$ is $\mathcal{K}$-copositive, then we take $\bar{h} = -h$, and we conclude in the same way that the solution sets of VIP$(\bar{F}, \Delta)$ and VIP$(\nabla \bar{h}, \Delta)$ coincide. Hence if B is $\mathcal{K}$-regular the solutions of CEiCP(B, C) are the stationary points for the problem of minimizing or maximizing h, given by (3.2), on $\mathcal{K}$ (where we minimize when B is $\mathcal{K}$-copositive and maximize when $-B$ is $\mathcal{K}$-copositive). We remark that, from a computational viewpoint, computing a stationary point of an optimization problem is in general much easier than finding a solution of a variational inequality problem. We also mention that in the case of $\mathcal{K} = \mathbb{R}^n_+$, the equivalence between solving EiCP and finding a stationary point of the Rayleigh quotient was established in [29].

This analysis also holds for CQEiCP(A, B, C) when A, B and C are symmetric and $A, -C$ are positive definite. Indeed, we have seen in Section 2 that QEiCP(A, B, C) reduces to solving CEiCP(D, G) and CEiCP(D, H) with, D, G, H as in (2.2)–(2.4). If we take $E = -C$, then symmetry of A, B and C implies symmetry of D, G and H, and so CQEiCP(A, B, C) can be solved by finding the stationary points related to the Rayleigh quotients for D, G and D, H. Since $C \in S_0'(K)$ because $-C$ is PD (see [9]), and A is PD, CQEiCP has solutions, as a consequence of Theorem 2.7. Note that this reduction of CQEiCP to a stationary point also holds if A is strictly $\mathcal{K}$-copositive and $-C$ is PD.

4 Numerical solution of the symmetric CEiCP and CQEiCP with a second-order cone

In Section 3 we showed that if B and C are symmetric matrices and B is positive definite, then any stationary point $\tilde{x} \neq 0$ of the function h defined by (3.2) on a convex self-dual cone $\mathcal{K}$ solves the symmetric CEiCP, and that this approach is also useful for CQEiCP when A, B and C are symmetric and $-C$ is positive definite.

In this section we consider first CEiCP when $\mathcal{K}$ is the Second-Order cone defined by (1.14) and (1.15) (SOCEiCP). We start by introducing the normalization constraint (1.16) that prevents $x = 0$ from being a feasible solution of the corresponding nonlinear program to be solved. Then we consider the maximization of the Rayleigh Quotient function on the set defined by the constraints

(1.14), (1.15) and (1.16), that is, the problem:

$$\begin{aligned} \text{NLP:} \quad \text{Minimize} \quad & h(x) = -\frac{x^t C x}{x^t B x} \\ \text{subject to} \quad & x \in \mathcal{K}, \\ & \sum_{i=1}^{r} x_0^i = 1. \end{aligned} \tag{4.1}$$

In the following, we will denote, as in Section 3, the feasible set of NLP (4.1) by $\Delta = \left\{ x \in \mathcal{K} : \sum_{i=1}^{r} x_0^i = 1 \right\}$. Hence, any stationary point $\bar{x}$ of the NLP (4.1) is a complementary eigenvector of SOCEiCP and $-h(\bar{x})$ is the associated complementary eigenvalue.

Now consider SOCQEICP, where A, B and C are symmetric matrices and $A, -C$ are PD. Due to the equivalence between CQEiCP(A, B, C) and CEiCP(D, G) with $E = -C$, a positive quadratic complementary eigenvalue for SOCQEiCP can be computed by finding the objective function value at a stationary point of the following NLP:

$$\begin{aligned} \text{Minimize} \quad & \frac{y^t B y + 2x^t C y}{y^t A y - x^t C x} \\ \text{s.t.} \quad & x \in \mathcal{K}, \quad y \in \mathcal{K}, \\ & \sum_{i=1}^{r} (x_0^i + y_0^i) = 1. \end{aligned} \tag{4.2}$$

A negative quadratic complementary eigenvalue can also be computed as a stationary point of NLP (4.2), with the matrix $-B$ replacing B in the objective function.

Since the NLP (4.2) can be written in the form of the NLP (4.1), with $2n$ and $2r$ instead of n and r respectively, we restrict our attention to the NLP (4.1).

Next, we discuss the use of the so-called Spectral Projected-Gradient (SPG) algorithm for computing a stationary point $\tilde{x}$ of NLP (4.1). As stated before, $h(\tilde{x})$ and $\tilde{x}$ are a complementary eigenvalue and a complementary eigenvector respectively for the symmetric Second-Order cone (SOCEiCP). The SPG algorithm is a feasible descent method, which means that in each iteration k the current point x_k is feasible, i.e., $x_k \in \Delta$, and is updated by using a descent direction for the function h and a positive stepsize.

At iteration k, the projected gradient search direction d_k is given by

$$d_k = P_\Delta(x_k - \eta_k \nabla h(x_k)) - x_k, \tag{4.3}$$

where $\eta_k > 0$, $\nabla h(x_k)$ represents the gradient of h at x_k, and $P_\Delta(y)$ denotes the projection of y on Δ. If $u_k = x_k - x_{k-1}$ and $v_k = \nabla h(x_k) - \nabla h(x_{k-1})$ satisfy $u_k^t v_k > 0$, the so called Spectral parameter

$$\eta_k = \frac{u_k^t u_k}{u_k^t v_k} \tag{4.4}$$

should be used. If $u_k^t v_k \leq 0$, then η_k should be a positive real number chosen according to [20]. Now, either $d_k = 0$ and x_k is a stationary point of h at x_k or x_k is updated by $x_{k+1} = x_k + \delta_k d_k$, where the stepsize $\delta_k \in (0, 1]$ is computed by the exact line-search technique discussed in [20]. As discussed in [4], the algorithm converges to a stationary point of h under reasonable hypotheses. Alternatively [10, 19], η_k can be computed by

$$\eta_k = \frac{u_k^t v_k}{v_k^t v_k}. \tag{4.5}$$

The steps of the SPG algorithm are described below.

Spectral Projection Algorithm (SPG)

Step 0. Let $\epsilon > 0$ be a tolerance, choose $x_0 \in \Delta$ and let $k := 0$.

Step 1. Compute d_k according to (4.3).
If $\|d_k\| < \epsilon$, terminate. The current vector x_k is a stationary point of h on Δ. Otherwise, compute the stepsize $\delta_k \in (0, 1]$ by an exact line-search.

Step 2. Update

$$x_{k+1} := x_k + \delta_k d_k$$

and return to Step 1 with $k := k + 1$.

Next, we focus our attention to the choice of the initial point and the computation of the gradient, search direction and the stepsize.

(1) *Initial Point*
The initial point $x_0 = (x^1, \ldots, x^r) \in \mathbb{R}^n$ with $x^i = (x_0^i, \bar{x}^i) \in \mathbb{R} \times \mathbb{R}^{n_i - 1}$, $i = 1, \ldots, r$, has the following components:

$$x_0^i = \frac{1}{r}, \quad \bar{x}^i = \frac{1}{r}\, e^s,$$

where e^s is a vector of the canonical basis and $s = \min\{i, n_i - 1\}$.

(2) *Computation of the gradient* $\nabla h(x)$

The gradient of the (negative) Rayleigh Quotient function h at x is given by (3.3).

(3) *Computation of the Projected-Gradient Direction* d

The projected gradient search direction at each iteration is given by (4.3). It is well-known, e.g., [16, Prop. 3.3], that the projection of an arbitrary vector onto the second-order cone $\mathcal{K}$ can be obtained in an explicit form. However, due to the presence of the normalization constraint (1.16), it is by no means obvious how the projection onto the set Δ can be computed. Nevertheless, it is possible to design a special purpose efficient algorithm for the computation of the projection that exploits the particular structure of the feasible set Δ.

Now we describe in detail the proposed algorithm for computing the projection. Let a point $u = (u^1, \ldots, u^r) \in \mathbb{R}^n$ with $u^i = (u_0^i, \bar{u}^i) \in \mathbb{R} \times \mathbb{R}^{n_i - 1}, i = 1, \ldots, r$, be given. Then the projection of u onto the set Δ is the unique solution of the convex optimization problem:

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{Minimize}} & \dfrac{1}{2} \displaystyle\sum_{i=1}^{r} \|x^i - u^i\|^2 \\ \text{subject to} & \|\bar{x}^i\| - x_0^i \leq 0, \; i = 1, \ldots, r, \\ & \displaystyle\sum_{i=1}^{r} x_0^i = 1. \end{array} \tag{4.6}$$

For finding the optimal solution of problem (4.6), first fix $x_0^i \geq 0$, $i = 1, \ldots, r$ arbitrarily, and consider the following optimization problem for each i:

$$\begin{array}{ll} \underset{\bar{x}^i \in \mathbb{R}^{n_i - 1}}{\text{Minimize}} & \dfrac{1}{2} \|x^i - u^i\|^2 \\ \text{subject to} & \|\bar{x}^i\| - x_0^i \leq 0. \end{array}$$

Noticing that $\|x^i - u^i\|^2 = (x_0^i - u_0^i)^2 + \|\bar{x}^i - \bar{u}^i\|^2$, it is not difficult to see that the optimal solution $\bar{x}^i$ of this problem is given by

$$\bar{x}^i = \begin{cases} \bar{u}^i & \text{if } x_0^i \geq \|\bar{u}^i\| \\ \dfrac{x_0^i}{\|\bar{u}^i\|} \bar{u}^i & \text{if } x_0^i < \|\bar{u}^i\|, \end{cases} \tag{4.7}$$

and the optimal value is given by

$$\phi_i(x_0^i|u^i) := \begin{cases} \frac{1}{2}(x_0^i - u_0^i)^2 & \text{if } x_0^i \geq \|\bar{u}^i\| \\ \frac{1}{2}(x_0^i - u_0^i)^2 + \frac{1}{2}(x_0^i - \|\bar{u}^i\|)^2 & \text{if } x_0^i < \|\bar{u}^i\|. \end{cases}$$

Thus the optimal solution of problem (4.6) is obtained by solving the following convex optimization problem with variables $x_0^i \in \mathbb{R}, i = 1, \ldots, r$:

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{r} \phi_i(x_0^i|u^i) \\ \text{subject to} \quad & \sum_{i=1}^{r} x_0^i = 1, \\ & x_0^i \geq 0, \ i = 1, \ldots, r. \end{aligned} \tag{4.8}$$

In the sequel, for the sake of a simpler notation, we denote $\phi_i(x_0^i)$ for $\phi_i(x_0^i|u^i)$, $i = 1, \ldots, r$. Note that the functions ϕ_i are strongly convex and continuously differentiable. More specifically, the first derivatives of ϕ_i are given by

$$\phi_i'(x_0^i) = \begin{cases} x_0^i - u_0^i & \text{if } x_0^i \geq \|\bar{u}^i\| \\ 2x_0^i - (u_0^i + \|\bar{u}^i\|) & \text{if } x_0^i < \|\bar{u}^i\|. \end{cases} \tag{4.9}$$

Observe that ϕ_i' is an increasing, piecewise linear and concave function for all i. More specifically, each ϕ_i' has two linear pieces and a single kink, where the right directional derivative is 1 and the left one is 2, which means $\lim_{t \to -\infty} \phi_i'(t) = -\infty$ and $\lim_{t \to \infty} \phi_i'(t) = \infty$.

Since problem (4.8) is convex, the following KKT conditions are necessary and sufficient for optimality:

$$\phi_i'(x_0^i) - v - w_i = 0, \quad i = 1, \ldots, r, \tag{4.10}$$

$$\sum_{i=1}^{r} x_0^i = 1, \tag{4.11}$$

$$x_0^i \geq 0, \ w_i \geq 0, \ x_0^i w_i = 0, \quad i = 1, \ldots, r, \tag{4.12}$$

where $v \in \mathbb{R}$ and $w_i \in \mathbb{R}$, $i = 1, \ldots, r$, are Lagrange multipliers.

From (4.10) and (4.12), we have

$$w_i = \phi_i'(x_0^i) - v \geq 0, \quad i = 1, \ldots, r,$$

which implies

$$x_0^i \geq (\phi_i')^{-1}(v), \quad i = 1, \ldots, r, \tag{4.13}$$

where $(\phi_i')^{-1}$ is the inverse function of ϕ_i', which is well-defined by the above-mentioned property of ϕ_i'. In fact, the function $(\phi_i')^{-1}$ has the following explicit representation for each i, cf. (4.9):

$$(\phi_i')^{-1}(v) = \begin{cases} v + u_0^i & \text{if } v \geq -(u_0^i - \|\bar{u}^i\|) \\ \frac{1}{2}(v + u_0^i + \|\bar{u}^i\|) & \text{if } v < -(u_0^i - \|\bar{u}^i\|). \end{cases}$$

Moreover, from (4.13) and the complementarity condition (4.12), we obtain

$$x_0^i = \max(0, (\phi_i')^{-1}(v)), \quad i = 1, \ldots, r, \tag{4.14}$$

which together with (4.11) yields the following univariate equation with variable $v \in \mathbb{R}$:

$$\sum_{i=1}^{r} \max(0, (\phi_i')^{-1}(v)) = 1. \tag{4.15}$$

Once the solution v of equation (4.15) is found, the solution x_0^i, $i = 1, \ldots, r$, of problem (4.8) is obtained from (4.14), which along with (4.7) then yields the desired solution $(x_0^i, \bar{x}^i)$, $i = 1, \ldots, r$, of problem (4.6).

Now let us show that the solution of equation (4.15) can be computed efficiently. To this end, it will be convenient to define the functions $\psi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, r$, by

$$\psi_i(v) = \max(0, (\phi_i')^{-1}(v))$$

and scalars $\alpha_i, \beta_i, i = 1, \ldots, r$, by

$$\begin{aligned} \alpha_i &:= -(u_0^i + \|\bar{u}^i\|), \\ \beta_i &:= -(u_0^i - \|\bar{u}^i\|). \end{aligned} \tag{4.16}$$

The numbers $-\alpha_i$ and $-\beta_i$ are nothing but the spectral values of vector u^i with respect to the second-order cone $\mathcal{K}_i$, see, e.g., [16]. Note that $\alpha_i \leq \beta_i$ for all i; moreover, $\alpha_i = \beta_i$ if and only if $\bar{u}^i = 0$. Then the functions ψ_i can be represented explicitly as follows:

- If $\alpha_i < \beta_i$, then

$$\psi_i(v) = \begin{cases} v + u_0^i & \text{if } v \geq \beta_i \\ \dfrac{1}{2}(v + u_0^i + \|\bar{u}^i\|) & \text{if } \alpha_i \leq v < \beta_i \\ 0 & \text{if } v < \alpha_i. \end{cases}$$

- If $\alpha_i = \beta_i$, then

$$\psi_i(v) = \begin{cases} v + u_0^i & \text{if } v \geq \alpha_i \\ 0 & \text{if } v < \alpha_i. \end{cases}$$

In any case, the functions ψ_i are piecewise linear and convex. The subgradients of these functions are given as follows:

- If $\alpha_i < \beta_i$, then

$$\partial\psi_i(v) = \begin{cases} \{1\} & \text{if } v > \beta_i \\ \left[\dfrac{1}{2}, 1\right] & \text{if } v = \beta_i \\ \left\{\dfrac{1}{2}\right\} & \text{if } \alpha_i < v < \beta_i \\ \left[0, \dfrac{1}{2}\right] & \text{if } v = \alpha_i \\ \{0\} & \text{if } v < \alpha_i. \end{cases}$$

- If $\alpha_i = \beta_i$, then

$$\partial\psi_i(v) = \begin{cases} \{1\} & \text{if } v > \alpha_i \\ [0, 1] & \text{if } v = \alpha_i \\ \{0\} & \text{if } v < \alpha_i. \end{cases}$$

Now let us define the function $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(v) = \sum_{i=1}^{r} \psi_i(v) - 1.$$

Then the equation (4.15) can be rewritten as

$$\varphi(v) = 0. \tag{4.17}$$

It is not difficult to see that $\varphi(v) = -1$ for all $v \leq \alpha$, where

$$\alpha := \min_{1 \leq i \leq r} \alpha_i$$

with α_i given by (4.16). Moreover, any $\xi \in \partial\varphi(v)$ is positive whenever $v > \alpha$, and hence φ is strictly increasing on (α, ∞), and $\lim_{v\to\infty} \varphi(v) = \infty$. Consequently, equation (4.17) has a unique solution $v^* \in (\alpha, \infty)$. Once v^* is computed, the optimal solution of problem (4.8) is obtained from (4.14) with $v = v^*$. Moreover, the optimal solution of problem (4.6), i.e., the projection of u onto $\mathcal{K}$, is recovered from (4.7) with x_0^i so obtained.

A number of algorithms are available for solving the univariate equation (4.17). Below we present a (generalized) Newton method.

Newton's method for solving equation (4.17).

Step 0 Find an initial solution v_0 such that $\varphi(v_0) > 0$. Let $k := 0$.

Step 1 If $\varphi(v_k) = 0$, then terminate. Otherwise, go to Step 2.

Step 2 Choose a subgradient $\xi_k \in \partial\varphi(v_k) = \partial\psi_1(v_k) + \cdots + \partial\psi_r(v_k)$, and compute v_{k+1} by

$$v_{k+1} = v_k - \frac{\varphi(v_k)}{\xi_k}. \tag{4.18}$$

Let $k := k + 1$ and go to Step 1.

Notice that $\xi_k > 0$ for any k, since $v_k > \alpha$. Hence the iteration (4.18) is well-defined. Moreover, since the function φ is monotonically increasing, piecewise linear (with a finite number of pieces) and convex, it can easily be shown that the method is finitely convergent to the unique solution v^*. For completeness, we prove this fact.

Proposition 4.1. Newton's method for solving equation (4.15) finds its unique solution v^* in a finite number of iterations.

Proof. First observe that

$$\varphi(v_{k+1}) \geq \varphi(v_k) + \xi_k(v_{k+1} - v_k) = 0, \quad k = 0, 1, 2, \ldots,$$

where the inequality and the equality hold from $\xi_k \in \partial\varphi(v_k)$ and (4.18), respectively. This means $v_k \geq v^* > \alpha$ for all k. Moreover, since $\xi_k > 0$, (4.18) implies $v_{k+1} < v_k$ whenever $\varphi(v_k) > 0$. Consequently, as long as $v_k > v^*$, $\{v_k\}$ is a strictly decreasing sequence bounded below by v^*.

By the afore-mentioned properties of φ, there exist a finite number of points $v^* \equiv s_0 < s_1 < \cdots < s_p \equiv v_0$ such that φ is linear with positive slope on each interval $[s_i, s_{i+1}]$, $i = 0, 1, \ldots, p-1$. If $v_k \in [v^*, s_1)$ for some k, we immediate have $v_{k+1} = v^*$ from (4.18), and the iteration terminates with the solution v^*. Now, suppose that the sequence $\{v_k\}$ is infinite and $v_k \geq s_1$ for all k. Then we have $\varphi(v_k) \geq \varphi(s_1) > 0$ and, by convexity, $\xi_k \geq \dfrac{\varphi(s_1) - \varphi(v^*)}{s_1 - v^*} > 0$ for all k. This along with (4.18) implies that the positive sequence $\{v_k - v_{k+1}\}$ is bounded away from zero, which contradicts the assumption that $\{v_k\}$ is bounded below by s_1. Hence we must have $v_k \in [v^*, s_1)$ for some k, and then $v_{k+1} = v^*$. This completes the proof. $\square$

Remark 4.2. (i) We need to find an initial solution v_0 such that $\varphi(v_0) > 0$. From a practical viewpoint, a small initial value v_0 is preferred, as long as it satisfies $\varphi(v_0) > 0$. Since $\varphi(\alpha) = -1$ and φ is monotonically increasing for $v > \alpha$, we may set $v_0 := \alpha + \hat{\ell}\delta$ for some $\delta > 0$, where $\hat{\ell}$ is the smallest positive integer ℓ such that $\varphi(\alpha + \ell\delta) > 0$.

(ii) In Step 1 we use the stopping criterion $|\varphi(v_k)| < \varepsilon$, with ε a small positive tolerance (in practice $\varepsilon = \sqrt{\bar{\epsilon}}$, where $\bar{\epsilon}$ is the machine precision).

(4) *Computation of the stepsize δ*

The value of the stepsize is obtained with an exact line-search, i.e., it is the solution of the univariate optimization problem

$$\begin{array}{ll} \text{Minimize} & g(\delta) \\ \text{subject to} & 0 \leq \delta \leq 1, \end{array}$$

where $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(\delta) = h(x + \delta d)$, for given vectors x and d. According to [20], any solution δ of $g'(\delta) = 0$ associated with the Rayleigh quotient function is a root of the following equation of degree two:

$$a_1 + \delta a_2 + \delta^2 a_3 = 0, \tag{4.19}$$

where

$$a_1 = (d^t C x)(x^t B x) - (d^t B x)(x^t C x),$$

$$a_2 = (d^t C d)(x^t B x) - (d^t B d)(x^t C x),$$

$$a_3 = (d^t Cd)(x^t Bd) - (d^t Bd)(x^t Cd).$$

Let s_1 and s_2 be the solutions of equation (4.19). Noticing that $\varphi'(0) < 0$ and $0 \leq \delta \leq 1$, we can determine the stepsize as

$$\delta = \begin{cases} 1 & \text{if } a_3 = 0 \text{ or } s_1, s_2 \notin [0,1] \\ s_i & \text{if } s_i \in [0,1], s_j \notin [0,1] \\ s_i & \text{if } s_1, s_2 \in [0,1] \text{ and } \varphi(s_i) \leq \varphi(s_j), \varphi(s_i) \leq \varphi(1) \\ 1 & \text{if } s_1, s_2 \in [0,1] \text{ and } \varphi(1) \leq \varphi(s_i) \ (i = 1, 2). \end{cases}$$

5 Computational experience

In this section we report some computational experience with the SPG algorithm discussed in the previous section for the solution of symmetric SOCEiCPs and SOCQEiCPs. The experiments have been performed on a Pentium IV (Intel) with 3.0 GHz and 2 GBytes of RAM memory, using the operating system Linux. The algorithm was coded in FORTRAN 90 and compiled with the GNU compiler, version 4.8.2. The algorithm was also implemented in the General Algebraic Modeling System (GAMS) language (Rev 227 x86_64/Linux) [8] and the solver MINOS [25] (Version 5.51) was used to solve the problem (4.1), where the constraints $\|\bar{x}^i\| \leq x_0^i$ were replaced by $\|\bar{x}^i\|^2 \leq (x_0^i)^2$. Running times presented in this section are always given in CPU seconds.

In our set of test problems, the matrix B of the SOCEiCP problem is the identity matrix and $C \in \mathbb{R}^{n \times n}$ is a symmetric matrix ($C = (E + E^t)/2$), where E is randomly generated such that each element is uniformly distributed in the interval $[-1, 1]$. For the SOCQEiCP instances, A is the identity matrix, $B = (E + E^t)/2$ is symmetric and $C = -(I_n + EE^t)$ (with I_n the identity matrix) is also symmetric such that $-C \in PD$. Furthermore, for the SPG algorithm the value of the stopping tolerance ϵ has been set to 1.0E-06 and the values of η_{min} and η_{max} have been fixed to 1.0E-05 and 1.0E+05, respectively.

Tables 1 and 2 report the results obtained with the SPG algorithm to solve SOCEiCPs and SOCQEiCPs respectively, considering 3 and 5 Lorentz cones (i.e., $r = 3$ and $r = 5$). In these experiments the spectral parameter η_k was computed by the formula (4.4). We also tested the parameter η_k given by the alternative formula (4.5) but the numerical results with this choice for η_k were in general worse than those obtained when formula (4.4) was used. In the following tables

IT is the total number of iterations, λ is the complementary eigenvalue computed, and T is the total CPU time in seconds required to solve each problem. We also solve all the test problems by the well-known code MINOS. The performance of this last method is also illustrated in Tables 1 and 2. The notations (1) and (2) stands for instances where the solver MINOS was not able to find a solution (solver found the problem unbounded or badly scaled or the solver was not able to terminate within the 3600 CPU seconds allowed, respectively). We denote by (3) the instance for which the SPG algorithm was not able to find a stationary point within the maximum of 10000 iterations allowed. It is important to notice that for this particular instance the algorithm stopped after 2.26E+02 CPU seconds of execution with $\|d\| = 2.48\text{E-}05$ at the last visited point. Furthermore, the algorithm required 1371 iterations to terminate with a stopping tolerance $\epsilon = 1.0\text{E-}04$.

Table 1: Performance of the algorithms to solve symmetric SOCEiCPs .

					SPG			MINOS		
r	n	n_1	n_2	n_3	IT	λ	T	IT	λ	T
	10	5	3	2	33	2.0491E+00	2.00E-04	56	2.3141E+00	1.40E-02
	20	10	5	5	40	2.8138E+00	5.00E-04	190	2.5457E+00	2.40E-02
	30	15	8	7	55	2.7716E+00	1.10E-03	119	2.7716E+00	2.60E-02
	40	20	10	10	97	2.7837E+00	1.90E-03	283	2.7695E+00	6.10E-02
	50	25	13	12	57	4.3995E+00	2.70E-03	141	4.3995E+00	5.50E-02
	60	30	15	15	77	4.6203E+00	4.00E-03	192	4.6203E+00	8.40E-02
	70	35	18	17	61	5.0735E+00	5.00E-03	207	5.0735E+00	1.20E-01
3	80	40	20	20	90	5.2576E+00	7.20E-03	195	5.2576E+00	1.46E-01
	90	45	23	22	82	5.5120E+00	8.80E-03	266	5.5120E+00	2.27E-01
	100	50	25	25	220	5.9158E+00	1.72E-02	371	5.8207E+00	3.41E-01
	200	100	50	50	347	8.7372E+00	8.88E-02	390	8.7372E+00	1.38E+00
	300	150	75	75	1598	1.0444E+01	7.06E-01	882	9.1407E+00	5.98E+00
	400	200	100	100	201	1.2547E+01	2.54E-01	674	1.2547E+01	8.92E+00
	500	250	125	125	145	1.3274E+01	3.40E-01	755	1.3274E+01	1.59E+01
	1000	500	250	250	168	1.9215E+01	1.73E+00	1416	1.9215E+01	1.23E+02
		$n_i,\ i=1,\dots,5$								
	10		2		31	2.0433E+00	2.00E-04	6.20E+01	1.9823E+00	1.50E-02
	20		4		35	3.2463E+00	5.00E-04	1.04E+02	3.0575E+00	1.80E-02
	30		6		77	2.5276E+00	1.10E-03	3.54E+02	2.3607E+00	4.70E-02
	40		8		144	3.2950E+00	2.40E-03	4.09E+02	3.1474E+00	7.70E-02
	50		10		137	3.7164E+00	3.60E-03	6.49E+02	3.2991E+00	1.53E-01
	60		12		377	4.3943E+00	9.30E-03	2.56E+02	4.0818E+00	1.03E-01
	70		14		74	4.3176E+00	5.30E-03	7.48E+02	4.1019E+00	2.99E-01
5	80		16		148	5.1492E+00	9.10E-03	3.94E+02	4.0405E+00	2.39E-01
	90		18		78	5.6044E+00	8.70E-03	1.28E+03	5.3459E+00	8.58E-01
	100		20		147	5.6063E+00	1.54E-02	8.17E+02	4.7623E+00	6.11E-01
	200		40		698	7.7786E+00	1.49E-01		(1)	
	300		60		689	9.5695E+00	3.40E-01	6.31E+02	8.7583E+00	4.48E+00
	400		80		137	1.1500E+01	2.08E-01		(1)	
	500		100		1473	1.2762E+01	1.83E+00	1.25E+03	1.1064E+01	2.35E+01
	1000		200		493	1.8729E+01	3.66E+00	1.67E+03	1.7119E+01	1.46E+02

The results shown in these tables demonstrate the efficiency and efficacy of the SPG algorithm

Table 2: Performance of the algorithms to solve SOCQEiCPs.

					SPG			MINOS		
r	n	n_1	n_2	n_3	IT	λ	T	IT	λ	T
3	10	5	3	2	84	3.0263E+00	1.00E-03	1.89E+02	3.0263E+00	1.56E-02
	20	10	5	5	114	4.3714E+00	5.00E-03	2.99E+02	3.8208E+00	3.00E-02
	30	15	8	7	172	6.6440E+00	1.10E-02	3.10E+02	6.6440E+00	5.00E-02
	40	20	10	10	1050	6.9840E+00	4.60E-02	1.86E+03	6.6058E+00	6.20E-01
	50	25	13	12	391	8.2448E+00	3.40E-02	2.98E+03	8.2448E+00	1.55E+00
	60	30	15	15	810	8.0960E+00	7.80E-02	1.05E+03	7.6763E+00	7.10E-01
	70	35	18	17	486	9.9073E+00	7.70E-02	1.08E+03	9.1118E+00	1.01E+00
	80	40	20	20	1784	9.3005E+00	2.76E-01	5.64E+03	7.3178E+00	7.30E+00
	90	45	23	22	690	1.1881E+01	1.61E-01	1.56E+04	1.1881E+01	2.85E+01
	100	50	25	25	1320	1.1437E+01	3.31E-01		(1)	
	200	100	50	50	1231	1.7206E+01	1.18E+00		(1)	
	300	150	75	75	2582	2.0557E+01	5.22E+00	2.42E+03	1.2650E+01	4.29E+01
	400	200	100	100	2249	2.3533E+01	9.06E+00		(1)	
	500	250	125	125	3802	2.6027E+01	2.33E+01		(1)	
	1000	500	250	250	6562	3.6311E+01	1.58E+02		(2)	
			$n_i,\ i = 1, \ldots, 5$							
5	10		2		78	2.7549E+00	1.00E-03	2.33E+03	4.2921E-01	1.26E-01
	20		4		218	4.2182E+00	6.00E-03	5.28E+02	4.2182E+00	8.60E-02
	30		6		165	6.1334E+00	1.00E-02	5.64E+03	4.7885E+00	1.23E+00
	40		8		739	6.7211E+00	3.50E-02	2.72E+03	6.2289E+00	1.04E+00
	50		10		565	7.6913E+00	4.20E-02	2.25E+03	6.9284E+00	1.25E+00
	60		12		1020	8.7394E+00	9.40E-02	8.63E+03	7.6240E+00	6.84E+00
	70		14		724	9.4442E+00	1.01E-01	8.24E+03	8.2050E+00	8.77E+00
	80		16		1260	1.0528E+01	2.04E-01		(1)	
	90		18		1221	1.0754E+01	2.59E-01		(1)	
	100		20		1147	1.0954E+01	2.93E-01		(1)	
	200		40		1273	1.6549E+01	1.20E+00		(1)	
	300		60		2464	2.1000E+01	4.89E+00		(1)	
	400		80		2922	2.4658E+01	1.15E+01		(1)	
	500		100		3603	2.6073E+01	2.22E+01		(1)	
	1000		200			(3)			(1)	

for solving the symmetric SOCEiCP and the SOCQEiCP. The projection technique described in the previous section has performed very well for all the instances. The performance of this projection technique and of the SPG algorithm does not seem to be influenced by an increase of the number r of the Lorenz cones $\mathcal{K}_i$.

In order to have a better idea of the efficiency of the SPG algorithm, we compared its performance with the solver MINOS. The SPG algorithm was able to solve all but one of the test problems with fewer iterations and less CPU time. We can conclude that the solver MINOS has difficulties to find a solution of SOCQEiCPs, particularly when the dimension increases. Furthermore, the SPG algorithm is usually very efficient to solve all the SOCEiCP and SOCQEiCP test problems and it performs in general better than MINOS. In fact, the number of iterations and CPU time are smaller for the SPG algorithm and the gap between the times tends to increase with the dimension of the problems.

6 Conclusions

In this paper, we discuss the existence of a solution to the Conic Quadratic Eigenvalue Complementarity Problem (CQEiCP), where the vectors x and w of complementary variables belong to an arbitrary pointed, closed and convex cone $\mathcal{K}$ and its dual $\mathcal{K}^*$. A sufficient condition for the existence of a solution for CQEiCP is introduced.

It is shown that, assuming that two of its defining matrices are PD, the symmetric CQEiCP reduces to the computation of a stationary point $\tilde{x} \neq 0$ of an appropriate merit function on a convex set. The numerical solution of the symmetric CEiCP and CQEiCP when $\mathcal{K}$ is the so called Second-Order Cone (SOCEiCP and SOCQEiCP respectively) by the Spectral Projected-Gradient (SPG) algorithm is also investigated. A new technique for computing projections required by the SPG method is introduced. The SPG method and the projection technique seem to perform very well in practice for solving the symmetric SOCEiCP and SOCQEiCP. The solution of the nonsymmetric SOCQEiCP is certainly one of our main research interests in the near future.

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