RESEARCH ARTICLE

On the Symmetric Quadratic Eigenvalue Complementarity Problem

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In this paper, the solution of the symmetric Quadratic Eigenvalue Complementarity Problem (QE-iCP) is addressed. The QEiCP has a solution provided the so-called co-regular and co-hyperbolic properties hold and is said to be symmetric if all the matrices involved in its definition are symmetric. We show that under the two conditions stated above the symmetric QEiCP can be reduced to the problem of computing a stationary point of an appropriate nonlinear program (NLP).

We also investigate the reduction of the QEiCP to a simpler Eigenvalue Complementarity Problem (EiCP). This transformation enables us to show that the co-regular and co-hyperbolic properties are not necessary for the existence of a solution to the QEiCP. Furthermore the QEiCP is shown to be equivalent to the problem of finding a stationary point of a Quadratic Fractional Program (QFP) under special conditions on the matrices of the QEiCP.

The use of the so-called Spectral Projected-Gradient (SPG) algorithm for dealing with the programs NLP and QFP is also investigated. Some considerations about the implementation of this algorithm are discussed. Computational experience is included to highlight the efficiency of the algorithm for finding a solution of the QEiCP by exploring the nonlinear programs mentioned above.

Keywords: Eigenvalue Problems, Complementarity Problems, Nonlinear Programming, Global Optimization.

AMS Subject Classification: 90B60, 90C33, 90C30, 90C26

1.   Introduction

Given a matrix $A \in \mathbb{R}^{n \times n}$ and a positive definite (PD) matrix $B \in \mathbb{R}^{n \times n}$ (i.e., $x^T B x > 0$ for all $x \neq 0$), the *Eigenvalue Complementarity Problem* (**EiCP**) [24, 26] consists of finding a real number λ and vectors $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$ such that

$$\textbf{EiCP: } w = (\lambda B - A)x \tag{1}$$

$$w \geq 0, x \geq 0 \tag{2}$$

$$x^T w = 0 \tag{3}$$

$$e^T x = 1, \tag{4}$$

where $e \in \mathbb{R}^n$ is a vector of ones. The last constraint has been introduced without loss of generality to prevent the null vector x from being a solution of the problem. The problem finds many applications in engineering [23, 26] and can be seen as a generalization of

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the well-known Eigenvalue Problem (**EiP**) [9]. As for the EiP, in any solution of the EiCP, the scalar λ is called an eigenvalue and x is an eigenvector associated to λ. The condition $x^T w = 0$ together with the nonnegative requirements on the variables x_i and w_i implies that $x_i = 0$ or $w_i = 0$ for each $i = 1, 2, \ldots, n$. These two variables are called complementary, giving the name [5, 6, 18] to the EiCP. It is known [14] that the EiCP always has a solution, as it can be reformulated as a Variational Inequality Problem on the simplex

$$\Omega = \{x \in \mathbb{R}^n : e^T x = 1, x \geq 0\}. \tag{5}$$

The existence of solutions of the EiCP is also guaranteed under the weaker hypothesis that B is Strictly Copositive (SC), that is, when $x^T B x > 0$ for all $0 \neq x \geq 0$ [14].

If the matrices A and B are both symmetric and B is PD, the EiCP is called symmetric and reduces to the problem of finding a Stationary Point (**SP**) of the so-called Rayleigh Quotient function on the simplex Ω [24, 29], that is, a SP of the following Standard Quadratic Fractional Program

$$\textbf{SQFP: Minimize } \frac{x^T A x}{x^T B x}$$
$$\text{subject to } e^T x = 1$$
$$x \geq 0.$$

A number of techniques have been proposed to solve the EiCP and its extensions [1, 4, 12–15, 19, 22, 28, 31]. As expected, the symmetric EiCP is easier to solve. A spectral projected-gradient algorithm has been proposed in [13] for this task. The structure of the SQFP is fully exploited for the computation of the gradient and of the projection required by the algorithm in each iteration. Furthermore it is possible to design an exact line-search for finding the stepsize in each iteration that essentially requires the solution of a binomial equation. Computational experience illustrates the efficiency of the algorithm for finding a solution for the symmetric EiCP [13].

Recently an extension of the EiCP has been introduced in [27], where some applications are highlighted. This so-called Quadratic Eigenvalue Complementarity Problem (QEiCP) differs from the usual EiCP on the existence of an additional quadratic term on λ and takes the following form

$$\textbf{QEiCP: } w = \lambda^2 A x + \lambda B x + C x$$
$$w \geq 0, x \geq 0$$
$$x^T w = 0$$
$$e^T x = 1,$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{n \times n}$ are given matrices and, as before, the constraint $e^T x = 1$ has been introduced without loss of generality to prevent the null vector to be a solution of the problem. Contrary to the EiCP, the QEiCP may have no solution even when the matrix A of the leading term is PD. This has led to the introduction of the co-regular and co-hyperbolic properties [27] given by

$$x^T A x \neq 0 \text{ for all } 0 \neq x \geq 0 \tag{6}$$
$$(x^T B x)^2 \geq 4(x^T A x)(x^T C x) \text{ for all } 0 \neq x \geq 0 \tag{7}$$

respectively. Under these two hypothesis, the QEiCP can be shown to be equivalent

to a Variational Inequality Problem on the simplex and has always a solution [27]. A number of algorithms has been proposed to solve the QEiCP. Among these techniques a semi-smooth Newton's method has been discussed in [27] and is in general fast for finding a solution to the QEiCP. However, the algorithm only possesses local convergence and may fail to achieve its goal. A line-search technique in the spirit of [6, 20] can be employed to compute a stationary point of some merit function [6]. However, there is no guarantee that such a stationary point is a solution of the QEiCP. Recognizing this difficulty, the enumerative method discussed in [14] has been extended in [7] to deal with the QEiCP. This algorithm is in general efficient for finding an solution to the QEiCP but may require too much tree search for some instances [7]. A hybrid method combining the best features of the semi-smooth and enumerative algorithms has also been recommended in [8] and seems to improve the efficiency of the enumerative method in practice.

In this paper, we investigate the solution of the symmetric QEiCP, that is, when all the matrices A, B and C are symmetric. As for the EiCP, we are able to show that if the co-regular and co-hyperbolic properties (6) and (7) hold then the symmetric QEiCP is equivalent to the problem of finding a stationary point of each one of the two merit functions

$$\lambda(x) = \frac{-x^T B x + \sqrt{(x^T B x)^2 - 4(x^T A x)(x^T C x)}}{2 x^T A x} \tag{8}$$

and

$$\bar{\lambda}(x) = \frac{-x^T B x - \sqrt{(x^T B x)^2 - 4(x^T A x)(x^T C x)}}{2 x^T A x} \tag{9}$$

on the simplex Ω given by (5).

We also analyze the reduction of the QEiCP to an EiCP. First, we show that if C is symmetric SC and $B = 0$ then the QEiCP has a solution if and only if there exists a vector $\bar{x} \geq 0$ such that $\bar{x}^T A \bar{x} < 0$. This property also implies that the co-regular and co-hyperbolic conditions are not necessary for the existence of a solution to the QEiCP. Furthermore we prove that for $C \in$ SC the QEiCP is equivalent to an augmented EiCP with eigenvectors (x, y) belonging to $\Omega \times \mathbb{R}^n_+$. If C is further symmetric and $A = -I$ then the QEiCP has a solution that can be obtained by finding a stationary point of the following Quadratic Fractional Program

$$\textbf{QFP: } \text{Maximize } f(x, y) = \frac{-x^T B x + 2 x^T y}{x^T C x + y^T y} \tag{10}$$

$$\text{subject to } x \geq 0, y \geq 0$$

$$e^T x = 1.$$

We also investigate the solution of the symmetric QEiCP when A is SC and no condition is assumed for C. In particular, we are able to show its reduction to the previous case.

As for the symmetric EiCP [13], the special structure of the merit functions (8), (9) and (10) and of the constraint sets Ω and $\Omega \times \mathbb{R}^n_+$ of the corresponding nonlinear programs has led us to investigate the use of the spectral projected-gradient method [2] for computing a stationary point of these functions. In this paper we discuss the possible use of this algorithm, namely the computation of the gradients and projections. For the solution of the QFP it is shown that the stepsize required by the projected-gradient algorithm

can be computed by an exact line-search that essentially requires the solution of a binomial equation. Some computational experience with medium and large scale QEiCP is included to illustrate the efficiency of the projected-gradient method for dealing with the symmetric QEiCP.

The organization of the paper is as follows. In Section 2, the merit functions (8) and (9) are introduced. The reductions of a QEiCP into an EiCP are discussed in Section 3. The projected-gradient method is described in Section 4. Computational experience with this algorithm is reported in Section 5. Finally some conclusions are presented in the last section of the paper.

2.   Nonlinear Programming Formulations for QEiCP

Consider again the QEiCP

$$w = \lambda^2 Ax + \lambda Bx + Cx \tag{11}$$

$$w \geq 0, x \geq 0 \tag{12}$$

$$x^T w = 0 \tag{13}$$

$$e^T x = 1, \tag{14}$$

and assume that the co-regular and co-hyperbolic properties (6) and (7) hold. Furthermore assume that the condition (7) is satisfied with a strict inequality. Then (11) and (13) imply

$$\lambda^2(x^T Ax) + \lambda(x^T Bx) + x^T Cx = 0. \tag{15}$$

If Ω is the simplex (5), then for each $x \in \Omega$ the roots of the binomial in λ (15) are given by (8) and (9), or equivalently by

$$\lambda(x) = -r(x) + \sqrt{(r(x))^2 - s(x)} \tag{16}$$

and

$$\bar{\lambda}(x) = -r(x) - \sqrt{(r(x))^2 - s(x)} \tag{17}$$

where

$$r(x) = \frac{x^T Bx}{2x^T Ax}, \qquad s(x) = \frac{x^T Cx}{x^T Ax}. \tag{18}$$

Under the hypotheses stated above, the functions are continuously differentiable on an open set containing Ω. Furthermore, as the co-regular property (6) holds then one of the matrices A or $-A$ must be strictly copositive (SC) and the following result can be established.

THEOREM 2.1

(i) If $A \in SC$, then any stationary point of

$$Maximize\ \lambda(x) \tag{19}$$

$$subject\ to\ x \in \Omega \tag{20}$$

and of

$$\text{Minimize } \bar{\lambda}(x) \tag{21}$$

$$\text{subject to } x \in \Omega \tag{22}$$

is a solution of QEiCP.

(ii) If $-A \in SC$*, then any stationary point of*

$$\text{Maximize } \bar{\lambda}(x) \tag{23}$$

$$\text{subject to } x \in \Omega \tag{24}$$

and of

$$\text{Minimize } \lambda(x) \tag{25}$$

$$\text{subject to } x \in \Omega \tag{26}$$

is a solution of QEiCP.

Proof We only prove the result for the nonlinear program (19)–(20), as the proofs for the remaining cases are similar. Consider the nonlinear program equivalent to (19)–(20)

$$\textbf{NLP: } \text{Minimize } -\lambda(x) \tag{27}$$
$$\text{subject to } x \in \Omega.$$

The KKT conditions for a Stationary Point of NLP (27) are given by

$$\nabla(-\lambda(x)) = \alpha e + w$$
$$w \geq 0, \ x \geq 0,$$
$$x^T w = 0,$$
$$e^T x = 1.$$

Note that

$$\nabla(-\lambda(x)) = \nabla r(x) - \frac{2r(x)\nabla r(x) - \nabla s(x)}{2\sqrt{(r(x))^2 - s(x)}}$$
$$= \frac{1}{2\sqrt{(r(x))^2 - s(x)}}[2\lambda(x)\nabla r(x) + \nabla s(x)].$$

Since

$$\nabla r(x) = \frac{1}{2} \frac{2[(x^T Ax)Bx - (x^T Bx)Ax]}{(x^T Ax)^2} = \frac{1}{x^T Ax}[Bx - 2r(x)Ax] \tag{28}$$

$$\nabla s(x) = \frac{1}{x^T Ax}[2Cx - 2s(x)Ax], \tag{29}$$

we have

$$2\lambda(x)\nabla r(x) + \nabla s(x) = \frac{1}{x^T A x}[2\lambda(x)Bx - 4\lambda(x)r(x)Ax + 2Cx - 2s(x)Ax]$$

$$= \frac{2}{x^T A x}[(-2\lambda(x)r(x) - s(x))Ax + \lambda(x)Bx + Cx].$$

Moreover

$$(\lambda(x))^2 = (r(x))^2 - 2r(x)\sqrt{(r(x))^2 - s(x)} + (r(x))^2 - s(x)$$

$$= -2r(x)\lambda(x) - s(x).$$

Hence we can write

$$\nabla(-\lambda(x)) = \frac{1}{(x^T A x)\sqrt{(r(x))^2 - s(x)}}[(\lambda(x))^2 Ax + \lambda(x)Bx + Cx].$$

Furthermore

$$x^T \nabla(-\lambda(x)) = 0$$

due to the definition of $\lambda(x)$. But

$$0 = x^T \nabla(-\lambda(x)) = \alpha(e^T x) + x^T w$$

implies $\alpha = 0$. Hence

$$\begin{cases} \nabla(-\lambda(x)) = w \\ x \geq 0, \ w \geq 0 \\ x^T w = 0 \\ e^T x = 1 \end{cases}$$

Therefore x and $\lambda(x)$ are an eigenvector and an eigenvalue for the QEiCP. ∎

This theorem shows that under the co-regular and co-hyperbolic properties (6) and (7) the QEiCP has a solution which can be found by computing a stationary point of one the merit functions (16) and (17) on the simplex (5).

3.   Reduction of QEiCP to EiCP

3.1.   *Case of $B = 0$ and $C \in SC$*

Consider again the QEiCP and assume that C is a Strictly Copositive (SC) matrix, that is,

$$\forall x \in \Omega \quad x^T C x > 0$$

where Ω is the simplex given by (5). Note that this class strictly contains the one of Positive Definite (PD) matrices, that is, those matrices C satisfying

$$\forall x \neq 0 \quad x^T C x > 0.$$

The definition of an SC matrix implies that $\lambda = 0$ can not be a solution of the QEiCP. Then for $C \in$ SC and $B = 0$, the QEiCP reduces to the following EiCP

$$w = \mu C x - (-A)x \tag{30}$$

$$w \geq 0, x \geq 0 \tag{31}$$

$$x^T w = 0 \tag{32}$$

$$e^T x = 1. \tag{33}$$

Furthermore $\mu > 0$ is an eigenvalue with corresponding eigenvector $\bar{x}$ for the EiCP (30)–(33) if and only if $\lambda = \pm\frac{1}{\sqrt{\mu}}$ are eigenvalues for the QEiCP associated to the same eigenvector $\bar{x}$. So the QEiCP reduces to the problem of finding a positive eigenvalue for the EiCP (30)–(33). The following result gives a necessary and a sufficient condition for the symmetric EiCP to have a positive eigenvalue.

THEOREM 3.1 *If A and C are symmetric matrices and $C \in$ SC, then EiCP (30)–(33) has a positive eigenvalue μ if and only if there exists an $\bar{x} \in \Omega$ such that $\bar{x}^T A \bar{x} < 0$.*

Proof (i) If $\mu > 0$ is an eigenvalue of EiCP (30)–(33), then it follows from (30) and (33) that an associated eigenvector $\bar{x}$ satisfies

$$\mu = \frac{\bar{x}^T(-A)\bar{x}}{\bar{x}^T C \bar{x}} > 0.$$

Since $\mu > 0$ and $\bar{x}^T C \bar{x} > 0$, $\bar{x}^T A \bar{x} < 0$.
    (ii) Consider the nonlinear program

$$\textbf{NLP: Maximize } \frac{x^T(-A)x}{x^T C x} = f(x) \tag{34}$$

subject to $x \in \Omega$.

Since there exists $\bar{x} \in \Omega$ such that $\bar{x}^T A \bar{x} < 0$, we have $f(\bar{x}) > 0$. On the other hand, Ω is a compact set and the NLP has a global maximum $\tilde{x}$, which is a stationary point of f on Ω. Hence [24]

$$\tilde{\mu} = \frac{\tilde{x}^T(-A)\tilde{x}}{\tilde{x}^T C \tilde{x}}$$

is an eigenvalue for the EiCP (30)–(33). Furthermore

$$\tilde{\mu} = \frac{\tilde{x}^T(-A)\tilde{x}}{\tilde{x}^T C \tilde{x}} \geq \frac{\bar{x}^T(-A)\bar{x}}{\bar{x}^T C \bar{x}} = f(\bar{x}) > 0.$$

∎

Note that the EiCP (30)–(33) can be solved by computing a stationary point of NLP (34) by a feasible ascent algorithm with an initial point $\bar{x}$ satisfying $\bar{x}^T A \bar{x} < 0$. Despite the computation of such a point being NP-hard [18], it is in general easy to find this point [13, 24]. For instance if A has a negative diagonal element a_{ii} then $\bar{x} = e^i$ can be this initial point, where e^i is the vector defined by $e^i_j = 1$ for $j = i$ and $e^i_j = 0$ otherwise.
    If the EiCP (30)–(33) is solved by computing a stationary point $\bar{x}$ of NLP (34), then $\frac{1}{\sqrt{f(\bar{x})}}$ and $-\frac{1}{\sqrt{f(\bar{x})}}$ are both eigenvalues of QEiCP associated to the same eigenvector

$\bar{x}$. This shows that if $B = 0$, A and C are symmetric and $C \in \mathrm{SC}$, then the QEiCP has at least one positive and one negative eigenvalues.

It is easy to see that this theorem does not hold if the matrix A is not symmetric. In fact, let C be the identity matrix of order 2 ($C \in \mathrm{SC}$) and

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then $(e^1)^T A e^1 = -1 < 0$, where $e^1 = (1, 0)^T \in \mathbb{R}^2$. However, it is easy to see that $\mu = -1$ is the unique eigenvalue of the EiCP (30)–(33).

Note that Theorem 3.1 also implies that the co-regular and co-hyperbolic properties (6) and (7) are not necessary for the QEiCP to have a solution. In fact, consider, the QEiCP with $B = 0$, C the identity matrix of order 2 and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then $C \in \mathrm{SC}$ and

$$(e^2)^T A e^2 < 0$$

where $e^2 = (0, 1)^T \in \mathbb{R}^2$. By Theorem 3.1, the QEiCP has at least a solution which satisfies the conditions

$$\lambda^2 x_1 + x_1 = w_1$$
$$-\lambda^2 x_2 + x_2 = w_2$$
$$w_1 x_1 = w_2 x_2 = 0$$
$$x_1 + x_2 = 1$$
$$x_i \geq 0, \quad w_i \geq 0, \quad i = 1, 2.$$

Then $x_1 = 0$, $x_2 = 1$ and $\lambda^2 = 1$. Hence $\bar{\lambda} = 1$ and $\tilde{\lambda} = -1$ are the eigenvalues of QEiCP. On the other hand $\tilde{x} = (1/2, 1/2)$ satisfies

$$\tilde{x}^T A \tilde{x} = 0$$

which means that the co-regular property is not satisfied. Furthermore for $e^1 = (1, 0)^T$,

$$(e^1)^T B e^1 - 4((e^1)^T A e^1)((e^1)^T C e^1) = -4.$$

Hence the co-hyperbolic property does not hold.

3.2.   Case of $B \neq 0$, $A = -I$ and $C \in SC$

Consider again the QEiCP (11)–(14). Since $C \in \mathrm{SC}$ $\lambda \neq 0$ in any solution of the QEiCP. Let us consider first the case of $\lambda > 0$. By introducing an additional vector y such that $y = \lambda x$, it is possible to write the equality (11) in the form

$$\begin{bmatrix} w \\ 0 \end{bmatrix} = \left(\lambda \begin{bmatrix} B & A \\ -I & 0 \end{bmatrix} + \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix}.$$

Since $\lambda > 0$, $\mu = \frac{1}{\lambda} > 0$ and we can consider the following EiCP:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \left(\mu \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} -B & -A \\ I & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} x \geq 0, &\quad u \geq 0 \\ y \geq 0, &\quad v \geq 0 \\ e^T x &= 1 \\ x^T u = y^T v &= 0. \end{aligned} \tag{35}$$

Next, we show that this EiCP is equivalent to the QEiCP (11)–(14).

Let $(\bar{x}, \bar{y}, \bar{\mu})$ be a solution of EiCP (35). If $\bar{y} = 0$, then $\bar{v} = -\bar{x} = 0$ which is impossible. Then there exists at least an i such that $\bar{y}_i > 0$. Now

(i) $\bar{y}_i > 0 \Rightarrow \bar{v}_i = 0 \Rightarrow \mu \bar{y}_i = x_i \Rightarrow \mu = \dfrac{\bar{x}_i}{\bar{y}_i}$.

(ii) $\bar{y}_i = 0 \Rightarrow \bar{v}_i = -x_i \Rightarrow \bar{v}_i = \bar{x}_i = 0$.

Hence $\bar{v} = 0$ in any solution of EiCP (35). Furthermore $e^T \bar{x} = 1, \bar{x} \geq 0$ and ($\bar{y}_i = 0 \Rightarrow \bar{x}_i = 0$), imply that $\bar{\mu} > 0$. So $(\bar{\lambda} = \frac{1}{\bar{\mu}}, \bar{x})$ is a solution of QEiCP. The converse follows immediately from the construction of the EiCP (35).

It is important to add that any eigenvalue $\bar{\mu}$ of EiCP (35) satisfies

$$\bar{\mu} = \frac{\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}^T \begin{bmatrix} -B & -A \\ I & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}}{\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}^T \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}} = \frac{\bar{x}^T (\bar{y} - A\bar{y} - B\bar{x})}{\bar{x}^T C \bar{x} + \bar{y}^T \bar{y}} \tag{36}$$

Furthermore the EiCP (35) is equivalent to the following Variational Inequality Problem (VI) [4]:

$$\textbf{VI: Find } \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \in \Omega \times \mathbb{R}_+^n : \tag{37}$$

$$F(\bar{x}, \bar{y})^T \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \right) \geq 0 \qquad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \Omega \times \mathbb{R}_+^n,$$

where Ω is given by (5) and $F \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is defined by

$$F(x, y) = \left(\frac{x^T (y - Ay - Bx)}{x^T C x + y^T y} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} -B & -A \\ I & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix}. \tag{38}$$

Since $\mathbb{R}_+^n$ is not a compact set, there is no guarantee that the VI (37) has a solution [6]. Necessary and sufficient conditions for the existence of a solution to this VI should be investigated in the future as they provide similar conditions for the existence of a solution to the QEiCP.

Now consider the case where B and C are symmetric matrices and $A = -I$. Then

$$\begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} -B & -A \\ I & 0 \end{bmatrix}$$

are symmetric matrices. The following theorem shows that in this case the QEiCP is equivalent to finding a stationary point of the nonlinear program QFP (10).

THEOREM 3.2 *If $A = -I$, B and C are symmetric matrices and $C \in SC$, then any stationary point of QFP (10) is a solution of QEiCP.*

Proof The KKT conditions for QFP (10) are given by

$$\frac{(2Bx - 2y)(x^T Cx + y^T y) - (x^T Bx - 2x^T y)(2Cx)}{(x^T Cx + y^T y)^2} = \alpha e + w \tag{39}$$

$$\frac{-2x(x^T Cx + y^T y) - 2y(x^T Bx - 2x^T y)}{(x^T Cx + y^T y)^2} = v \tag{40}$$

$$x \geq 0, \ w \geq 0, \ x^T w = 0$$

$$y \geq 0, \ v \geq 0, \ y^T v = 0$$

$$e^T x = 1,$$

where $\alpha \in \mathbb{R}$, $v \in \mathbb{R}^n$, and $w \in \mathbb{R}^n$ are the Lagrange multipliers associated to the constraints $e^T x = 1$, $y \geq 0$ and $x \geq 0$ respectively.

If $\mu = f(x, y)$ then the conditions (39) and (40) can be rewritten as follows

$$\frac{2}{x^T Cx + y^T y}(\mu Cx + Bx - y) = \alpha e + w \tag{41}$$

$$\frac{2}{x^T Cx + y^T y}(\mu y - x) = v. \tag{42}$$

Since $y \geq 0$, $v \geq 0$, $x \geq 0$, $e^T x = 1$, then (42) implies $\mu > 0$. Furthermore, for each $i = 1, \ldots, n$

$$v_i = 0 \Rightarrow \mu y_i = x_i.$$

On the other hand, if $v_i > 0$, then $y_i = 0$ and $x_i < 0$. Hence v_i cannot be positive and $\mu y_i = x_i$ for all $i = 1, \ldots, n$.

Multiplying (41) by $\frac{1}{2}(x^T Cx + y^T y)x^T$, we obtain

$$\mu x^T Cx + x^T Bx - y^T x = \frac{\alpha}{2}(x^T Cx + y^T y). \tag{43}$$

But since

$$\mu = \frac{-x^T Bx + 2x^T y}{x^T Cx + y^T y} = \frac{-x^T Bx + \frac{2}{\mu}x^T x}{x^T Cx + \frac{1}{\mu^2}x^T x},$$

we have

$$1 = \frac{-\mu x^T Bx + 2x^T x}{\mu^2 x^T Cx + x^T x}.$$

Then

$$\mu^2 x^T C x + x^T x + \mu x^T B x - 2 x^T x = 0,$$

which along with $\mu y = x$ and $\mu > 0$ yields

$$\mu x^T C x + x^T B x - y^T x = 0.$$

Then (43) implies $\alpha = 0$.

Substituting $p = 0$ in (41), the KKT conditions take the following form

$$\mu C x + B x - y = p$$
$$\mu y = x$$
$$x \geq 0, \ p \geq 0, \ x^T p = 0$$
$$e^T x = 1.$$

By writing $\mu p = q$, we obtain

$$q = \mu^2 C x + \mu B x - x$$
$$x \geq 0, \ q \geq 0, \ x^T q = 0$$
$$e^T x = 1.$$

Hence x is a solution of QEiCP with $\lambda = \dfrac{1}{\mu}$.

∎

Note that $f(x, y)$ cannot be unbounded along a recession direction $(\Delta x, \Delta y)$ of the feasible set. In fact, such a direction satisfies

$$\Delta x = 0, \ \Delta y \geq 0 \ \ (\Delta y \neq 0)$$

and for any feasible solution $\bar{x} \in \Omega, \bar{y} \in \mathbb{R}^n_+$,

$$f(\bar{x} + \alpha \Delta x, \bar{y} + \alpha \Delta y)$$
$$= \frac{-\bar{x}^T B \bar{x} + 2\bar{x}^T (\bar{y} + \alpha \Delta y)}{\bar{x}^T C \bar{x} + (\bar{y} + \alpha \Delta y)^T (\bar{y} + \alpha \Delta y)}$$
$$= \frac{-\bar{x}^T B \bar{x} + 2\bar{x}^T \bar{y} + 2\alpha \, \bar{x}^T \Delta y}{\bar{x}^T C \bar{x} + \bar{y}^T \bar{y} + 2\alpha \, \bar{y}^T \Delta y + \alpha^2 (\Delta y)^T \Delta y} \to 0$$

as $\alpha \to +\infty$. Therefore a stationary point $(\bar{x}, \bar{y})$ of QFP (10) exists and $(\bar{\lambda} = \frac{1}{f(\bar{x}, \bar{y})}, \bar{x})$ is a solution of the QEiCP.

Now consider the case of a negative eigenvalue for the QEiCP. Then (11) can be written as $w = (-\lambda)^2 A x + (-\lambda)(-B)x + C x$. If B and C are symmetric, $C \in$ SC and $A = -I$, then the QEiCP can be solved by computing a stationary point of the QFP (10) with $-B$ instead of B. Hence the following result holds.

THEOREM 3.3 *If $A = -I$, B and C are symmetric matrices and $C \in SC$, then QEiCP has at least a positive and a negative eigenvalue.*

Furthermore these eigenvalues can be computed as stationary points of quadratic fractional programs. Note that the existence of an eigenvalue for the QEiCP for $A = -I$ and $C \in$ SC is also guaranteed independently of the symmetry of B or C, by the fact that the co-regular and co-hyperbolic properties hold.

3.3. *Case of $A \in$ SC*

Consider the homogeneous Linear Complementarity Problem (LCP):

$$w = Cx, \quad x \geq 0, \quad w \geq 0, \quad x^T w = 0. \tag{44}$$

Recall [5] that C is called an R_0-matrix if and only if the LCP (44) has the unique solution $x = w = 0$. Then the following result holds independently of the SC condition on A.

THEOREM 3.4 *QEiCP has a solution $\lambda = 0$ if and only if $C \notin R_0$.*

Proof QEiCP has a solution $\lambda = 0$ if and only if the Linear Complementarity Problem

$$
\begin{aligned}
w &= Cx \\
x &\geq 0, \ w \geq 0 \\
x^T w &= 0 \\
e^T x &= 1
\end{aligned}
$$

has a solution, and this means that $C \notin R_0$.   ■

Suppose that $A \in$ SC and QEiCP has a solution. If $C \notin R_0$ then QEiCP has the solution $\lambda = 0$. Otherwise, QEiCP may have a solution $\lambda > 0$ or $\lambda < 0$. If $\lambda > 0$, we can write the QEiCP as follows

$$
\begin{aligned}
\frac{1}{\lambda^2} w &= Ax + \frac{1}{\lambda} Bx + \frac{1}{\lambda^2} Cx \\
x &\geq 0, \ w \geq 0 \\
x^T w &= 0 \\
e^T x &= 1
\end{aligned}
$$

and the QEiCP reduces to the case of Subsection 3.2. If $\lambda < 0$ the by writing $\bar{\lambda} = -\lambda$ the QEiCP can be written as

$$
\begin{aligned}
\frac{1}{\bar{\lambda}^2} w &= Ax + \frac{1}{\bar{\lambda}} (-B)x + \frac{1}{\bar{\lambda}^2} Cx \\
x &\geq 0, \ w \geq 0 \\
x^T w &= 0 \\
e^T x &= 1
\end{aligned}
$$

and again reduces to the case of Subsection 3.2.

As a conclusion, the symmetric EiCP can always be reduced to the problem of finding a stationary point of an appropriate merit function when $A \in$ SC and $C = -I$.

4.   A Projected-Gradient Algorithm

In Section 2 it has been shown that if the co-regular and co-hyperbolic properties (6) and (7) hold, then the symmetric QEiCP can be solved by computing a stationary point of one of the nonlinear programs introduced in Theorem 2.1. The constraint set of these programs is the simplex Ω defined by (5). The special structure of this set Ω makes the computation of projections of vectors over Ω very easy. On the other hand, the objective functions of the required nonlinear programs have Hessians whose computation is quite involved. These features lead to our decision of investigating first order algorithms that are based on gradients and projections. We have chosen the so-called Spectral Projected-Gradient (SPG) algorithm [2] mainly due to its good performance for solving the symmetric EiCP by way of similar nonlinear programs [13]. In this section we discuss this algorithm not only for dealing with the nonlinear programs mentioned before but also with the QFP (10).

In order to explain the SPG algorithm, let us consider a nonlinear program of the form

$$\text{Minimize } h(x) \tag{45}$$
$$x \in X$$

where h is a C^1 function on an open subset of $\mathbb{R}^n$ containing the feasible set X. The SPG algorithm is a feasible descent method, which means that in each iteration k the current point x_k is feasible, i.e., $x_k \in X$ and is updated by using a descent direction for the function h and a positive stepsize [20]. If $x_k \in X$ is the current point the so-called projected-gradient direction d_k is computed by

$$d_k = P_X(x_k - \eta_k \nabla h(x_k)) - x_k \tag{46}$$

where $\eta_k > 0$, $\nabla h(x_k)$ represents the gradient of h at x_k and $P_X(z)$ denotes the projection of $z \in \mathbb{R}^n$ on X. If $u_k = x_k - x_{k-1}$ and $v_k = \nabla h(x_k) - \nabla h(x_{k-1})$ satisfy $u_k^T v_k > 0$, the so-called Spectral parameter [2, 25]

$$\eta_k = \frac{u_k^T u_k}{v_k^T u_k} \tag{47}$$

should be used. In case of $v_k^T u_k \le 0$, η_k should be a positive real number chosen according to [13]. Now, either $d_k = 0$ and x_k is a stationary point of h at x_k or a stepsize $\delta_k \in (0, 1]$ is computed by a line-search technique [6, 20]. The new iterate is given by $x_{k+1} = x_k + \delta_k d_k$ and satisfies $x_{k+1} \in X$ and $h(x_{k+1}) < h(x_k)$. A new iteration with x_{k+1} should be performed. The steps of the SPG algorithm are presented below.

SPG Algorithm

Step 0 - Let $\epsilon > 0$ be a tolerance, $x_0 \in X$ and $k = 0$.

Step 1 - Compute d_k by (46).

Step 2 - If $\|d_k\| < \epsilon$, stop with x_k a Stationary Point of h on X.

Step 3 - Compute $\delta_k \in (0, 1]$ by a line-search technique.

Step 4 - Update x_k by $x_{k+1} = x_k + \delta_k d_k$ and return to Step 1 with $k = k + 1$.

The SPG algorithm possesses global convergence to a stationary point of h on X under reasonable hypotheses [2]. Therefore the algorithm is able to find a solution of the symmetric QEiCP by computing a stationary point of one of the programs mentioned in Theorem 2.1 and of the QFP (10) when $A = -I$ and $C \in$ SC. Next, we discuss the choice of the initial point and the computation of gradients, search directions and stepsizes for each one of these nonlinear programs. We only consider the program

$$\textbf{NLP: } \text{Minimize } \lambda(x)$$
$$\text{subject to } e^T x = 1, \tag{48}$$
$$x \geq 0,$$

among those introduced in Theorem 2.1, as the application of the SPG to the remaining programs is similar. Furthermore QFP (10) is considered in the following equivalent form

$$\textbf{QFP: } \text{Minimize } \frac{x^T B x - 2 x^T y}{x^T C x + y^T y} = -f(x, y)$$
$$\text{subject to } e^T x = 1, \tag{49}$$
$$x \geq 0, \; y \geq 0.$$

4.1. *Initial Point*

As discussed in [13], when applied to NLP (48), the initial point for the SPG algorithm should be chosen by one of the two possibilities

$$x_0 = \frac{1}{n} e \tag{50}$$

$$x_0 = e^i \tag{51}$$

where $e \in \mathbb{R}^n$ is a vector of ones and e^i is a vector of the canonical basis of $\mathbb{R}^n$. For the QFP (49), x_0 should be chosen by (50) or (51) and

$$y_0 = \sigma x_0 \tag{52}$$

where σ is a nonnegative real number. In the next section we discuss the importance of these choices on the efficiency of the SPG algorithm.

4.2. *Computation of Gradients*

For NLP (48), the proof of Theorem 2.1 gives the following expression for the gradient $\nabla \lambda(x)$ at x:

$$\nabla \lambda(x) = \frac{-1}{2\sqrt{[r(x)]^2 - s(x)}} \left[2\lambda(x) \nabla r(x) + \nabla s(x) \right] \tag{53}$$

where $\nabla r(x)$ and $\nabla s(x)$ are given by (28) and (29), respectively. By simple algebraic manipulations, it is possible to design the following procedure for the computation of $\nabla \lambda(x)$:

Procedure Gradient of λ at $\bar{x}$: $g = \nabla \lambda(\bar{x})$

$$\begin{cases} u = A\bar{x} \\ v = B\bar{x} \\ z = C\bar{x} \\ \theta_1 = \bar{x}^T u \\ \theta_2 = \bar{x}^T v \\ \theta_3 = \bar{x}^T z \\ r = \frac{\theta_2}{2\theta_1}, \quad s = \frac{\theta_3}{\theta_1}, \quad \bar{\lambda} = \sqrt{(r)^2 - s}, \quad \lambda = -r + \bar{\lambda} \\ g = -\frac{1}{\theta_1 \lambda}[(\lambda)^2 u + \lambda v + z] \end{cases}$$

Now consider the QFP (49). Then, by simple linear algebra manipulations, (39) and (40) lead to the following procedure for computing the gradient of $-f$ at $(\bar{x}, \bar{y})$:

Procedure Gradient of $-f$ **at** $(\bar{x}, \bar{y})$: $g = (g_x, g_y) = -\nabla f(\bar{x}, \bar{y})$:

$$\begin{cases} u = B\bar{x} \\ v = C\bar{x} \\ \theta_1 = \bar{x}^T u \\ \theta_2 = \bar{x}^T v \\ \theta_3 = \bar{x}^T \bar{y} \\ \theta_4 = \bar{y}^T \bar{y} \\ \lambda_1 = \theta_1 - 2\theta_3, \quad \lambda_2 = \theta_2 + \theta_4, \quad \phi = \frac{\lambda_1}{\lambda_2} \\ (g_x, g_y) = \frac{2}{\lambda_2}(u - y - \phi v, \, -\phi y - x) \end{cases}$$

As a conclusion, at each iteration the computation of the gradients for the two objective functions of NLP (48) and QFP (49) essentially requires matrix-vector products with the matrices of the QEiCP.

4.3.   *Computation of the Projected-Gradient Directions*

Consider again the NLP (48). Then in each iteration k, the projected-gradient direction is given by

$$d_k = P_\Omega(x_k - \eta_k \nabla \lambda(x_k)) - x_k \tag{54}$$

where Ω is the simplex defined by (5) and $P_\Omega(v)$ is the unique global minimum (and stationary point) of the Strictly Convex Separable Quadratic Program

$$\textbf{SCSQP: } \text{Minimize } \frac{1}{2}\|v - u\|_2^2 = \frac{1}{2}(v - u)^T(v - u)$$
$$\text{subject to } e^T u = 1, \tag{55}$$
$$u \geq 0.$$

As discussed in [13], the SCSQP can be solved by a number of algorithms [11, 21, 30]. Among these methods, the block principal pivoting algorithm [11] has proven to be very efficient [4, 13] and is used in the implementation of the SPG algorithm.

For the QFP (49), the projected search direction is given by

$$d_k = \begin{bmatrix} d_k^x \\ d_k^y \end{bmatrix} = \begin{bmatrix} P_\Omega(x_k + \eta_k \nabla_x f(x_k, y_k)) \\ P_{\mathbb{R}_+^n}(y_k + \eta_k \nabla_y f(x_k, y_k)) \end{bmatrix} \tag{56}$$

where $\nabla f(x_k, y_k) = [\nabla_x f(x_k, y_k), \nabla_y f(x_k, y_k)]$ can be computed by the procedure discussed before. Furthermore the projection over Ω is computed as explained before and $u = P_{\mathbb{R}_+^n}(v)$ is given by

$$u_i = \begin{cases} v_i, \text{ if } v_i \geq 0 \\ 0, \text{ otherwise} \end{cases} \tag{57}$$

for $i = 1, \ldots, n$.

Finally the spectral parameter η_k is computed according to the procedure discussed in [13].

4.4. *Line-Search*

For the NLP (48), the so-called Armijo Criterion [6, 20] is recommended and consists of finding a stepsize δ_k such that

$$\lambda(x_k + \delta_k d_k) \leq \lambda(x_k) + \delta_k \beta \nabla \lambda(x_k)^T d_k \tag{58}$$

where β is a small positive constant (usually $\beta = 10^{-4}$). In practice, δ_k is computed by a finite number of trials of the form $\frac{1}{\theta^r}$, $r = 0, 1, \ldots$, where θ is a real number greater or equal to 2 (usually $\theta = 2$).

It is well-known that the exact line-search is usually advantageous over an approximate one based on the Armijo Criterion or similar [20]. However the exact line-search is very difficult to implement for a general nonlinear function, as it requires finding the global minimum of the problem

$$\text{Minimize } \lambda(x_k + \delta d_k)$$

$$\text{subject to } \delta \geq 0.$$

Next, we show that such a technique can be efficiently implemented for the QFP. Consider the objective function $-f(x, y)$ of QFP. If $(\bar{x}, \bar{y})$ is the current point and $d = (d^x, d^y)$ is the search direction, then the line-search function $\varphi(\alpha)$ is given by

$$\varphi(\alpha) = -f(\bar{x} + \alpha d^x, \bar{y} + \alpha d^y) = \frac{(\bar{x} + \alpha d^x)^T B(\bar{x} + \alpha d^x) - 2(\bar{x} + \alpha d^x)^T(\bar{y} + \alpha d^y)}{(\bar{x} + \alpha d^x)^T C(\bar{x} + \alpha d^x) + (\bar{y} + \alpha d^y)^T(\bar{y} + \alpha d^y)}$$

$$= \frac{t_1 \alpha^2 + t_2 \alpha + t_3}{u_1 \alpha^2 + u_2 \alpha + u_3}$$

where

$$t_1 = (d^x)^T(Bd^x) - 2(d^x)^T d^y$$

$$t_2 = 2[(d^x)^T(B\bar{x}) - \bar{x}^T d^y - \bar{y}^T d^x]$$

$$t_3 = \bar{x}^T(B\bar{x}) - 2\bar{x}^T \bar{y}$$

$$u_1 = (d^x)^T(Cd^x) + (d^y)^T d^y$$

$$u_2 = 2[(d^x)^T(C\bar{x}) + \bar{y}^T d^y]$$

$$u_3 = \bar{x}^T(C\bar{x}) + \bar{y}^T \bar{y}.$$

Note that $B\bar{x}$, $C\bar{x}$, t_3 and u_3 have already been obtained in Step 1 of the SPG algorithm when the gradient of $-f$ at $(\bar{x}, \bar{y})$ is computed according to the Procedure Gradient of $-f$ at $(\bar{x}, \bar{y})$.

Now

$$\varphi'(\alpha) = \frac{(2t_1\alpha + t_2)(u_1\alpha^2 + u_2\alpha + u_3) - (2u_1\alpha + u_2)(t_1\alpha^2 + t_2\alpha + t_3)}{(u_1\alpha^2 + u_2\alpha + u_3)^2}$$

$$= \frac{v_1\alpha^2 + v_2\alpha + v_3}{(u_1\alpha^2 + u_2\alpha + u_3)^2}$$

where

$$v_1 = 2t_1u_2 + u_1t_2 - 2t_2u_1 - t_1u_2 = t_1u_2 - t_2u_1$$

$$v_2 = 2t_1u_3 + t_2u_2 - 2u_1t_3 - t_2u_2 = 2(t_1u_3 - u_1t_3)$$

$$v_3 = t_2u_3 - u_2t_3.$$

Hence

$$\varphi'(\alpha) = 0 \iff v_1\alpha^2 + v_2\alpha + v_3 = 0.$$

Let

$$\Delta = v_2^2 - 4v_1v_3.$$

Then there are the following cases:

(i) $\Delta < 0 \Rightarrow$ there is no root for $\varphi'(\alpha) = 0$. Since the search direction is descent, then $\bar{\alpha} = 1$.

(ii) $\Delta \geq 0 \Rightarrow$ Let s_i, $i = 1, 2$ (may be equal) be the roots of $\varphi'(\alpha) = 0$. Then
   - $\forall s_i \notin [0, 1] \Rightarrow \bar{\alpha} = 1$.
   - $\exists s_i \in [0, 1] \Rightarrow \bar{\alpha} = \arg\min\{\varphi(1), \varphi(s_i) : s_i \in [0, 1]\}$

Hence the exact line-search for QFP essentially reduces to the computation of the roots of a binomial equation.

5. Computational Experience

In this section, we report some computational experience with the projected-gradient method (SPG), discussed in Section 4, and the well-known code MINOS [17] for the computation of eigenvalues and eigenvectors to the QEiCP. All the tests have been

performed on a Pentium IV (Intel) with Hyperthreading, 3.0 GHz CPU, 2GB RAM computer, using the operating system Linux. The SPG algorithm was implemented in FORTRAN 77 with version 10 of the Intel FORTRAN compiler [10]. The solver MINOS was implemented in the General Algebraic Modeling System (GAMS) language (Rev 118 Linux/Intel) [3].

We considered a set of test problems, where A was taken as $-I$, with I the identity matrix, and B and C are symmetric matrices from the Harwell-Boeing Collection [16]. Furthermore C is a positive definite matrix and then $C \in SC$. Hence, these QEiCPs can be solved by computing stationary points for the nonlinear programs NLP (48) and QFP (49). These test problems are denoted by MatHHarwell(n), where n represents the order of the matrices A, B, and C. A second set of test problems, MatRHarwell(n), was constructed. This differs from the first set in the matrix B, which is given by $B = F^T F$, where the matrix F was randomly generated with elements uniformly distributed in the interval $(0,1)$. Furthermore the matrices B and C of the two sets of test problems have been scaled so that all their elements belong to the interval $[0,1]$. Note that the matrices B of the set MatHHarwell(n) are quite sparse while the matrices B of the set MatRHarwell(n) are dense. In our computational experience for solving the NLP (48), we considered two initial points given by (50) and (51), where i is such that

$$|b_{ii}| = \max\{|b_{jj}|\colon j = 1, 2, \ldots, n\}. \tag{59}$$

For the QFP (49), the initial vector x_0 is chosen and

$$y_0 = \begin{cases} 0, & \text{if } x_0 \text{ is given by (50)} \\ |b_{ii}|e^i, & \text{otherwise} \end{cases}$$

where, as before, i is such that (59) holds. The computational performance of the algorithms SPG and MINOS for solving the test problems mentioned above are displayed in Tables 1, 2, 3 and 4, where the following notation is used:

- $\epsilon-$ tolerance for the SPG algorithm;
- $\lambda-$ value of the eigenvalue computed by the algorithms;
- IT - number of iterations required by the algorithms;
- CPU - CPU time in seconds spent by the algorithms;
- c - exponent of the value 10^{-c} of the complementarity gap $x^T w$, for the eigenvalue λ and eigenvector x computed by the algorithms, where

$$w = \lambda^2 Ax + \lambda Bx + Cx; \tag{60}$$

- r - exponent of the value 10^{-r} of the smallest component of w given by (60), where x and λ are the eigenvector and eigenvalues computed by the algorithms;
- $*-$ SPG algorithm was not able to compute an eigenvalue and an eigenvector within 40 000 iterations.

Note that λ and x should be considered to be an eigenvalue and an eigenvector, respectively, if c and r are big, i.e., 10^{-c} and 10^{-r} are small. The bigger c and r are the better precisions of the eigenvalue and eigenvector are. Furthermore c and r equal to ∞ means that the precision is optimal, as $10^{-c} = 0$, i.e., $10^{-c} < \epsilon_M$, with ϵ_M the machine precision. Furthermore MINOS was only tested with the initial point given by (51) and (59), since such a choice is much better than (50) for a code based on basic and superbasic solutions [17].

The numerical results presented in the Tables 1, 2, 3 and 4 lead to the following observations:

Table 1.   Performance of SPG and MINOS for solving QFP (49) with initial point given by (51) and (59).

Problem	SPG - $\epsilon = 10^{-4}$					SPG - $\epsilon = 10^{-5}$					MINOS				
	λ	c	r	IT	CPU	λ	c	r	IT	CPU	λ	c	r	IT	CPU
MatHHarwell(48)	0.550	9	6	834	0.03	0.550	11	7	1290	0.05	0.654	16	4	106	0.01
MatHHarwell(66)	0.471	9	6	1194	0.06	0.473	11	8	3414	0.17	0.410	16	2	452	0.02
MatHHarwell(112)	0.316	9	5	4675	0.46	0.316	11	7	11595	1.14	0.316	∞	4	16	0.05
MatHHarwell(132)	0.416	9	7	3	0.00	0.416	11	7	4	0.00	0.416	16	5	46	0.08
MatHHarwell(153)	0.371	9	5	450	0.07	0.371	11	7	2514	0.41	0.371	17	3	1088	0.11
MatHHarwell(420)	0.317	10	7	44	0.03	0.317	12	7	58	0.05	0.318	17	4	19	0.93
MatRHarwell(48)	1.491	16	16	2	0.00	1.491	16	16	2	0.00	1.491	14	14	3	0.01
MatRHarwell(66)	1.315	16	16	2	0.00	1.315	16	16	2	0.00	1.315	17	17	4	0.03
MatRHarwell(112)	1.093	16	16	2	0.01	1.093	16	16	2	0.01	1.093	15	15	2	0.08
MatRHarwell(132)	1.092	17	17	2	0.02	1.092	17	17	2	0.02	1.092	15	15	3	0.11
MatRHarwell(153)	1.156	16	16	2	0.03	1.156	16	16	2	0.03	1.156	17	17	4	0.15
MatRHarwell1(420)	1.631	∞	∞	2	0.19	1.631	∞	∞	2	0.19	1.631	13	13	3	1.50

Table 2.   Performance of SPG for solving QFP (49) with initial point given by (50).

Problem	SPG - $\epsilon = 10^{-4}$					SPG - $\epsilon = 10^{-5}$				
	λ	c	r	IT	CPU	λ	c	r	IT	CPU
MatHHarwell(48)	0.550	9	6	372	0.01	0.550	11	7	688	0.03
MatHHarwell(66)	0.471	9	6	1483	0.08	0.474	11	8	3763	0.21
MatHHarwell(112)	0.316	10	5	432	0.04	0.316	11	6	8396	0.82
MatHHarwell(132)	0.326	9	5	2580	0.31	0.326	11	7	13838	1.69
MatHHarwell(153)	0.371	9	5	1079	0.17	0.371	11	7	3079	0.51
MatHHarwell(420)	0.316	10	6	983	0.81	0.316	12	7	4245	3.51
MatRHarwell(48)	1.437	15	15	14	0.00	1.437	15	15	14	0.00
MatRHarwell(66)	1.486	14	14	14	0.00	1.486	14	14	14	0.00
MatRHarwell(112)	1.558	10	10	13	0.01	1.558	16	16	14	0.01
MatRHarwell(132)	1.516	11	11	13	0.02	1.516	16	16	14	0.02
MatRHarwell(153)	1.502	11	11	13	0.03	1.502	∞	∞	14	0.03
MatRHarwell1(420)	1.548	10	10	12	0.20	1.548	16	16	13	0.20

Table 3.   Performance of SPG and MINOS for solving NLP (48) with initial point given by (51) and (59).

Problem	SPG - $\epsilon = 10^{-4}$					SPG - $\epsilon = 10^{-5}$					MINOS				
	λ	c	r	IT	CPU	λ	c	r	IT	CPU	λ	c	r	IT	CPU
MatHHarwell(48)	1.092	16	16	1	0.00	1.092	16	16	1	0.00	1.092	17	17	1	0.01
MatHHarwell(66)	0.475	∞	∞	1294	0.12	0.475	∞	∞	1294	0.12	0.410	17	2	193	0.02
MatHHarwell(112)	1.106	17	∞	1	0.01	1.106	17	∞	1	0.01	1.106	17	∞	1	0.06
MatHHarwell(132)	0.416	17	17	1	0.02	0.416	17	17	1	0.02	0.416	17	5	20	0.08
MatHHarwell(153)	0.383	18	4	796	0.40	*	*	*	*	*	0.371	∞	3	462	0.12
MatHHarwell(420)	1.245	16	∞	1	0.20	1.245	16	∞	1	0.20	1.245	17	∞	1	0.90
MatRHarwell(48)	1.491	16	16	1	0.00	1.491	16	16	1	0.00	1.491	16	16	1	0.02
MatRHarwell(66)	1.315	16	16	1	0.00	1.315	16	16	1	0.00	1.315	17	17	1	0.03
MatRHarwell(112)	1.093	17	17	1	0.01	1.093	17	17	1	0.01	1.093	16	16	1	0.09
MatRHarwell(132)	1.092	17	17	1	0.02	1.092	17	17	1	0.02	1.092	17	17	1	0.13
MatRHarwell(153)	1.156	16	16	1	0.03	1.156	16	16	1	0.03	1.156	17	17	1	0.18
MatRHarwell1(420)	1.631	∞	∞	1	0.20	1.631	∞	∞	1	0.20	1.631	16	16	1	1.71

Table 4.   Performance of SPG and MINOS for solving NLP (48) with initial point given by (50).

Problem	SPG - $\epsilon = 10^{-4}$					SPG - $\epsilon = 10^{-5}$				
	λ	c	r	IT	CPU	λ	c	r	IT	CPU
MatHHarwell(48)	0.550	17	4	7493	0.47	*	*	*	*	*
MatHHarwell(66)	0.459	18	3	14	0.00	*	*	*	*	*
MatHHarwell(112)	0.319	17	3	4	0.00	*	*	*	*	*
MatHHarwell(132)	0.326	17	6	206	0.03	0.326	17	7	2399	0.26
MatHHarwell(153)	0.371	∞	∞	11	0.02	0.371	∞	∞	11	0.02
MatHHarwell(420)	0.318	18	5	13677	46.36	*	*	*	*	*
MatRHarwell(48)	0.727	17	17	2	0.00	0.727	17	17	2	0.00
MatRHarwell(66)	1.195	∞	∞	2	0.00	1.195	∞	∞	2	0.00
MatRHarwell(112)	0.754	17	17	2	0.01	0.754	17	17	2	0.01
MatRHarwell(132)	0.920	16	16	2	0.02	0.920	16	16	2	0.02
MatRHarwell(153)	0.886	∞	∞	2	0.03	0.886	∞	∞	2	0.03
MatRHarwell1(420)	1.473	∞	∞	2	0.20	1.473	∞	∞	2	0.20

(i) The SPG algorithm with a tolerance of $\epsilon = 10^{-4}$ is always able to find an eigenvalue and an eigenvector when applied to both the nonlinear programs NLP (48) and QFP (49). Furthermore for the NLP (48), the SPG algorithm may face difficulties to terminate if the tolerance is chosen smaller than 10^{-4} (see Table 4, $\epsilon = 10^{-5}$).

(ii) The numerical precision of the eigenvalue and eigenvector computed by the SPG algorithm is good even for tolerance of 10^{-4} and usually better than that obtained by MINOS with its default parameters.

(iii) The choice (51) with (59) of a canonical vector e^i for the initial point seems to be more appropriate than using the barycenter (50).

(iv) The QFP (49) usually provides solutions with better precision than the NLP (48). This is expected, as the expression of the objective function of QFP (49) is simpler and does not involve square roots. However, it is important to add that the QFP formulation only applies to the case where $A = -I$ and $C \in$ SC, while NLP (48) seems to have a broader application.

(v) The SPG algorithm usually requires less computational time than MINOS, particularly when the dimension n of the QEiCP increases.

(vi) The MINOS and SPG algorithms may compute different eigenvalues and eigenvectors.

In conclusion, the SPG algorithm seems to be a valid approach for solving the symmetric QEiCP in practice by computing a stationary point of an appropriate nonlinear program.

6.   Conclusions

In this paper the solution of the symmetric Quadratic Eigenvalue Complementary Problem (QEiCP) is investigated. Two nonlinear programming formulations for the QEiCP are introduced and it is shown that stationary points of these programs provide eigenvalues and eigenvectors for the QEiCP. The use of the Spectral Projected-Gradient (SPG) algorithm for dealing with these nonlinear programs is investigated. Numerical evidence of the good performance of the SPG algorithm in practice is shown.

In this paper, it is also established that the so-called co-regular and co-hyperbolic properties do not have to hold in order to guarantee a solution to the symmetric QEiCP. The study of weaker sufficient conditions for the existence of a solution to the QEiCP should also be investigated in future, together with special purpose algorithms to deal with these problems in practice.

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