COMPLEMENTARY EIGENVALUES OF GRAPHS

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Abstract. In this paper, we study the Eigenvalue Complementarity Problem (EiCP) when its matrix A belongs to the class $S(G) = \{A = [a_{ij}] : a_{ij} = a_{ji} \neq 0 \text{ iff } ij \in E\}$, where $G = (V, E)$ is a connected graph. It is shown that if all non diagonal elements of $A \in S(G)$ are non positive, then A has a unique complementary eigenvalue, which is the smallest eigenvalue of A. In particular, zero is the unique complementary eigenvalue of the Laplacian and the normalized Laplacian matrices of a connected graph. The number $\mathfrak{c}(G)$ of complementary eigenvalues of the adjacency matrix of a connected graph G is shown to be bounded above by the number $\mathfrak{b}(G)$ of induced non isomorphic connected subgraphs of G. Furthermore, $\mathfrak{c}(G) = \mathfrak{b}(G)$ if the Perron roots of the adjacency matrices of these subgraphs are all distinct. Finally, the maximum number of complementary eigenvalues for the adjacency matrices of graphs is shown to grow faster than any polynomial on the number of vertices.

Keywords: Eigenvalue Problem; Complementarity Problem; Graphs; Spectral Graph Theory.
MSC: 05C50

1. Introduction

Given a real matrix A of order n, the Eigenvalue Complementarity Problem (EiCP) consists of finding a real number λ and a vector $x \in \mathbb{R}^n - \{0\}$ such that

$$w = Ax - \lambda x \tag{1}$$

$$x \geq 0, \ w \geq 0 \tag{2}$$

$$x^T w = 0 \tag{3}$$

where $w \in \mathbb{R}^n$. Each solution (λ, x) of EiCP satisfies the feasibility conditions (1) and (2) and the condition (3). Since x and w are nonnegative vectors then this last condition is equivalent to n conditions:

$$x_i w_i = 0, \ i = 1, \dots, n \tag{4}$$

So, for each i at most one of the variables x_i or w_i may be positive. These variables are called complementary [7, 9] and the constraint (3) is named the complementarity condition. If $w = 0$ and x is not required to be nonnegative then EiCP reduces to the well-known Eigenvalue Problem (EiP) [14]:

$$Ax = \lambda x \tag{5}$$

Therefore, EiCP is an extension of EiP that contains a complementarity condition (3) on nonnegative variables. In each solution (λ, x) of EiCP, λ is called a complementary eigenvalue and x is an associated complementary eigenvector. Furthermore EiCP (1) — (3) is said to be symmetric if its matrix A is symmetric. The words Pareto eigenvalue and Pareto eigenvector have been used by many authors to name these complementary eigenvalue and eigenvector (see for instance [31]).

EiCP was introduced in [31] and finds many applications in several areas of science, engineering and economics [1, 9, 28, 29]. A number of efficient algorithms have been designed for the solution of EiCP and some of its extensions for the symmetric and the nonsymmetric cases. Like the EiP, the symmetric EiCP is much easier to solve as it reduces to the computation of a stationary point of the so-called Rayleigh Quotient [14] on the ordinary simplex [30]. A local nonlinear programming solver [27] can be employed to find a solution of the symmetric EiCP. In particular, projected-gradient and DC algorithms have been proposed to solve efficiently the symmetric EiCP [20, 23, 30]. Semi-smooth [1, 2], projected [5, 29], DC [26] and enumerative [10, 21] algorithms have been proposed for the solution of the nonsymmetric EiCP and some of its extensions. Some of these methods can be combined in order to enhance their efficiency [5, 11].

Many applications of EiCP are modeled as undirected graphs and the matrix involved is often a matrix associated with a graph. The main goal of this paper is to provide tools for the EiCP that takes into account the structure of the matrix, when it is associated with a graph[1]. Given a graph G, among many other matrices, we emphasize here its adjacency matrix, its Laplacian matrix and its normalized Laplacian matrix (see next section for definitions). Spectral Graph Theory is the research area where the spectrum of a matrix is used to establish structural properties of its graph. The first papers in the area appeared in connection with quantum chemistry by Hückel [18] relating eigenvalues of the adjacency matrix with stability and energy of molecules. Out of the many applications of the adjacency matrix, we mention a few in which classical parameters are related to the spectra of graphs. The papers of Wilf [34] and Hofmann [17] provided bounds for the chromatic number of a graph in terms of the eigenvalues of the adjacency matrix. Important relations between the spectrum of the adjacency matrix and hamiltonicity of a graph G have been studied by several authors (see [13] for recent results).

A well known property due to Fiedler [12] about the spectrum of the Laplacian matrix of a graph G is that the multiplicity of zero as an eigenvalue equals the number of connected components of G. Hence the second smallest eigenvalue – called algebraic connectivity of G – is nonzero iff G is connected. The matrix tree theorem, attributed to Kirchhoff, establishes an important relation between the number of spanning trees and the nonzero Laplacian eigenvalues of a graph. These and many of the applications of EiP for the Laplacian matrix in combinatorial optimization, physics, chemistry and computer science, as well as mathematical properties and applications in graph theory itself, can be found in the surveys [24, 25] and references therein. More recently, the normalized Laplacian matrix has been defined in the context of random walks [6].

It is well-known that the symmetric EiP has exactly n real eigenvalues while the nonsymmetric EiP has at most n. During the past several years, some researchers have investigated the maximum number of complementary eigenvalues for the symmetric and nonsymmetric EiCP [29, 30, 31, 33, 35].

It has been shown that EiCP has always a finite number of complementary eigenvalues. In particular, the maximum number of complementary eigenvalues of a symmetric EiCP is $2^n - 1$ and this maximum is attained by a special matrix [29]. This number is even bigger for the nonsymmetric EiCP [33]. An enumerative algorithm for computing all the complementary eigenvalues have been proposed in [11] and has been shown to be efficient if the dimension n of the EiCP is not too large.

In this paper, we investigate the number of complementary eigenvalues of symmetric matrices that are associated with a graph G. We start in section 2 by reviewing some

[1]In this paper we always assume that the graph is undirected.

definitions, and introducing the class of matrices

$$S(G) = \{A = [a_{ij}] : A \text{ is symmetric and } a_{ji} \neq 0 \text{ iff } ij \in E\}, \tag{6}$$

where $G = (V, E)$ is a connected graph. In section 3 we establish that if all non diagonal elements of $A \in S(G)$ are non positive, then A has a unique complementary eigenvalue, which is the smallest eigenvalue of A. In particular, zero is the unique complementary eigenvalue of the Laplacian and the normalized Laplacian matrices of a connected graph G. In Section 4 we show that the complementary eigenvalues of the adjacency matrix of a graph G are the Perron roots of the adjacency matrices of the induced non isomorphic connected subgraphs of G. Hence, we show that for graphs G with n vertices, the number $\mathfrak{c}(G)$ of distinct complementary eigenvalues of its adjacency matrix is bounded above by the number $\mathfrak{b}(G)$ of these subgraphs. Furthermore $\mathfrak{c}(G) = \mathfrak{b}(G)$ if all the adjacency matrices of the nonisomorphic induced subgraphs of G have different Perron roots.

Finding $\mathfrak{b}(G)$ for a given graph G is a well-known hard problem in classical graph theory. Our result $\mathfrak{c}(G) \leq \mathfrak{b}(G)$ may be seen as a practical lower bound for $\mathfrak{b}(G)$, if we are able to compute all the distinct complementary eigenvalues of the adjacency matrix of G. An algorithm has been introduced in [11] for such a purpose and seems to perform well when the number n of the order of the EiCP matrix (i.e., number of vertices of the graph) is small. On the other hand, the number $\mathfrak{c}(G)$ of distinct complementary eigenvalues can be estimated for graphs where $\mathfrak{b}(G)$ can be computed. In Sections 5, 6 and 7, we investigate some properties of $\mathfrak{b}(G)$ and the estimation of the number of distinct complementary eigenvalues $\mathfrak{c}(G)$ for some graphs G by studying $\mathfrak{b}(G)$. In particular we show that both quantities $\mathfrak{c}(G)$and $\mathfrak{b}(G)$ may grow faster than any polynomial in the number n of vertices of G.

2. Matrices of a graph

Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E. The *Adjacency matrix* $A = A(G)$ of G is a square matrix of order n whose entries are

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

For the diagonal degree matrix D, where the entry d_{ii} is the degree of the vertex v_i, the matrix

$$L(G) = D - A \tag{8}$$

is the *Laplacian matrix* of G. The *normalized Laplacian matrix* $\mathcal{L}(G)$ of G is defined by

$$\mathcal{L}(i,j) = \begin{cases} 1, & \text{if } i = j \text{ and } d_i > 0, \\ -\frac{1}{\sqrt{d_i \cdot d_j}}, & \text{if } \{v_i, v_j\} \in E, \\ 0, & \text{otherwise,} \end{cases} \tag{9}$$

where d_i denotes the degree of $v_i \in V$. We observe that $\mathcal{L}(G) = D^{-1/2} L D^{-1/2}$, if G has no isolated vertices. In [4] the *perturbed Laplacian matrix* of the graph G was introduced as $\mathcal{P}\!\!\!\mathcal{L}(G) = D - A(G)$ where D is an arbitrary diagonal matrix and $A(G)$ is a weighted adjacency matrix. We notice that $L(G)$, $A(G)$ *and* $\mathcal{L}(G)$ are instances of the perturbed Laplacian matrix. In fact, $\mathcal{P}\!\!\!\mathcal{L}(G)$ encompasses other matrices associated with G.

An even more general look at matrices of a graph was observed in [19]. If $A = [a_{ij}]$ is a symmetric matrix of order n, the *graph of* A, $G = G(A)$, is the graph of n vertices entirely determined by the nondiagonal entries of A, so that G has an edge incident to vertices v_i and v_j (for $i \neq j$) if and only if $a_{ij} \neq 0$. We notice that we may represent the graph of any symmetric matrix as a graph with weights on its edges (determined by the nondiagonal elements). The diagonal entries of A may be seen as weights on its vertex set. In this context, for a given graph G, we may associate the set $S(G)$ of all matrices A whose graph is G, defined by (6).

Example 1. *Consider the following matrix M:*

$$M = \begin{bmatrix} 2 & 1 & -1 & \sqrt{2} & 0 \\ 1 & 3 & 0 & 4 & -\sqrt{3} \\ -1 & 0 & 4 & 0 & 0 \\ \sqrt{2} & 4 & 0 & -1 & 0 \\ 0 & -\sqrt{3} & 0 & 0 & 0 \end{bmatrix}$$

The weighted graph associated with M is given in Figure 1. The numbers inside the vertices represent their weights (diagonal elements).

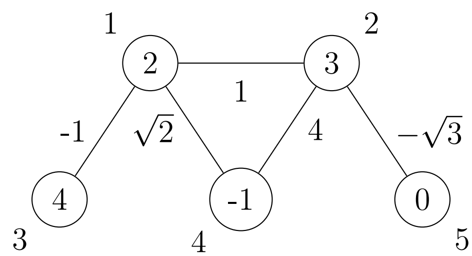

FIGURE 1. The graph of the matrix M.

3. UNIQUENESS OF COMPLEMENTARY EIGENVALUES

We start with a technical result that we state for completeness.

Lemma 1. [31] *Let A be a real matrix of order n and (λ, x) a solution to the EiCP* (1) – (3). *Then λ is an eigenvalue of a principal submatrix of A.*

Proof. We notice that a solution (λ, x) satisfies (4). Now consider

$$J = \{i : x_i > 0\} \text{ and } L = \{i : x_i = 0\}.$$

Then $w_J = 0 = A_{JJ}x_J + A_{JL}x_L - \lambda x_J = A_{JJ}x_J - \lambda x_J$, implying that

$$A_{JJ}x_J = \lambda x_J. \tag{10}$$

Hence λ is an eigenvalue of the principal submatrix A_{JJ} of A. □

Theorem 1. *Let G be a connected graph with n vertices and $S(G)$ be the set of matrices of G, defined by (6). If the non diagonal entries of $A \in S(G)$ are non positive, then A has a unique complementary eigenvalue, which is the smallest eigenvalue of A.*

Proof. As shown in [21, 31] EiCP has at least a solution. Let λ be such a complementary eigenvalue of A, and let x be the corresponding complementary eigenvector. Furthermore, let $J = \{i : x_i > 0\}$ and $L = \{i : x_i = 0\}$. By Lemma 1, λ is an eigenvalue of the principal submatrix A_{JJ} of A. Suppose that J is strictly contained in $\{1,\ldots,n\}$. Then we may write $x=(x_J,0)^T$ and x_J satisfies

$$\begin{aligned} w_J &= A_{JJ}x_J-\lambda x_J=0 && (11)\\ w_L &= A_{LJ}x_J\geq 0. && (12)\end{aligned}$$

Since the nondiagonal elements of A corresponding to edges are nonpositive and G is connected, there exists at least a pair (j,ℓ), with $j\in J$ and $\ell\in L$, such that $a_{\ell j}<0$. Hence (12) cannot hold and none of the eigenvalues of principal submatrices A_{JJ} of A with J strictly included in $\{1,\ldots,n\}$ can be complementary eigenvalues. Hence λ and x satisfy

$$Ax=\lambda x,\ x>0. \tag{13}$$

In order to prove that λ is the unique complementary eigenvalue, let $\overline{\lambda}$ be another complementary eigenvalue and $\overline{x}$ a corresponding complementary eigenvector. Therefore

$$A\overline{x}=\overline{\lambda}\overline{x},0\neq\overline{x}\geq 0. \tag{14}$$

Since A is a symmetric matrix, then

$$\begin{aligned} x^TA\overline{x} &= \overline{\lambda}x^T\overline{x}\\ x^TA\overline{x} &= \overline{x}^TAx=\lambda\overline{x}^Tx=\lambda x^T\overline{x}\end{aligned}$$

By (13) and (14), $x^T\overline{x}>0$ and $\overline{\lambda}=\lambda$. It remains to prove that λ is the smallest eigenvalue of A. The complementary eigenvector x associated to λ is a stationary point of

$$\min\ u^TAu\ \text{ s.t. } ||u||=1,\quad u\geq 0,$$

where $||u||$ denotes the Euclidean norm of u [32]. Furthermore, $\lambda=x^TAx$. Since $x>0$ by (14), then the inequalities $u\geq 0$ are all inactive at x and x is a stationary point of

$$\min\ u^TAu \text{ s.t. } ||u||=1.$$

Hence λ is the smallest eigenvalue of A [14]. □

As a consequence of this theorem and the fact that the zero is the smallest eigenvalue of $L(G)$ and $\mathcal{L}(G)$, the following result follows.

Theorem 2. *Zero is the unique complementary eigenvalue of the Laplacian and of the normalized Laplacian matrices of a connected graph.*

4. The adjacency matrix: a problem reduction

In this section we study the EiCP (1) – (3) when A is the adjacency matrix given by (7) of a graph $G=(V,E)$ with vertex set $V=\{v_1,\ldots,v_n\}$ and edge set E. For any solution (λ,x) of the EiCP,

$$0=x^Tw=x^TAx-\lambda x^Tx.$$

Hence

$$\lambda=\frac{x^TAx}{x^Tx}. \tag{15}$$

Since A is the adjacency matrix of a graph G, then $A\geq 0$, and this implies that λ is nonnegative. $\lambda=0$ is always a complementary eigenvalue associated to a canonical basis vector. Next, we will seek properties of positive complementary eigenvalues. We notice that a solution (λ,x) satisfies (4). So we can consider $J=\{i:x_i\ >\ 0\}$, and $L\ =\ \{i:x_i=0\}$. Then (11) and (12) hold and

$$A_{JJ}x_J = \lambda x_J, \tag{16}$$

implying that λ is an eigenvalue of the principal submatrix A_{JJ}. There is no loss of generality in assuming that A_{JJ} is irreducible, and, therefore, by the Perron-Frobenius theorem [14], there is a positive eigenvalue associated with a positive eigenvector.

Hence, each principal submatrix A_{JJ} of A gives at least a solution (λ, x) to the EiCP, with a positive eigenvalue given by the Perron root of A_{JJ}. By using a proof similar to the one presented for Theorem 1, we come to the conclusion that the Perron root of A_{JJ} is the unique complementary eigenvalue of EiCP associated to this principal submatrix. Hence we can state the following theorem.

Theorem 3. *Let G be a connected graph and A be its adjacency matrix. Let λ be the Perron root of A.*

(a) *λ is a complementary eigenvalue associated to a positive eigenvector.*
(b) *λ is the unique eigenvalue of A that is a complementary eigenvalue.*
(c) *All complementary eigenvalues are nonnegative.*
(d) *The positive complementary eigenvalues are the Perron roots of principal submatrices of A.*

We notice that each principal submatrix of an adjacency matrix corresponds to an *induced subgraph* of G, which is a subgraph where the edges are removed only when its vertices are.

We now observe that when a graph is disconnected, its spectrum is the union of the spectrum of each connected component. Then the Perron root is the largest of the Perron roots of its connected components. So in order to count the number of distinct Perron roots, it is enough to consider connected graphs and, more importantly, the counting may be done considering only connected (distinct) subgraphs.

Consequently, as a result of item (d) of the above proposition, the number of distinct complementary eigenvalues of the adjacency matrix of a graph is related to the number of distinct induced connected subgraphs. Since distinct subgraphs may have the same Perron roots, the latter is at least as large as the former.

We call the complementary eigenvalues of G the solutions to the EiCP associated to A. We summarize these conclusions in the following result.

Theorem 4. *Let G be a connected graph with n vertices and A its adjacency matrix. Let $\mathfrak{c}(G)$ be the number of complementary eigenvalues of G and $\mathfrak{b}(G)$ be the number of induced non isomorphic connected subgraphs of G. Then*

$$\mathfrak{c}(G) \leq \mathfrak{b}(G).$$

We formulate the question of interest in this note.

Question 1. *Given a graph G with n vertices, what is the number $\mathfrak{c}(G)$ of distinct complementary eigenvalues of G?*

By Theorem 4 $\mathfrak{c}(G)$ is bounded above by the number $\mathfrak{b}(G)$ of non isomorphic connected induced subgraphs of G. Furthermore $\mathfrak{c}(G) = \mathfrak{b}(G)$ if the adjacency matrices of all these subgraphs have distinct Perron roots. Unfortunately, the computation of $\mathfrak{b}(G)$ is a well known hard problem in graph theory (see, for example [3, 16]).

Since A is a symmetric matrix, then [31]

$$\mathfrak{c}(G) \leq 2^n - 1, \tag{17}$$

for a graph with n vertices. Theorem 4 shows that the inequality (17) is strict, as $\mathfrak{b}(G)$ is smaller than $2^n - 1$ (see Theorem 5 below). So the adjacency matrix of a graph is one of the classes of symmetric matrices such that the large upper bound given in (17) cannot be attained. So, it will be interesting to obtain an upper bound for $\mathfrak{c}(G)$ and $\mathfrak{b}(G)$ or at least to investigate whether $\mathfrak{b}(G)$ and $\mathfrak{c}(G)$ are exponential in n or not. This will be discussed in the next sections. Theorem 4 can also be used to estimate $\mathfrak{b}(G)$ by using $\mathfrak{c}(G)$. An enumerative method has been designed in [11] to compute all the complementary eigenvalues of a given matrix. This algorithm seems to perform well for matrices of small orders, whence it can be used to estimate $\mathfrak{b}(G)$ for graphs with small number of nodes.

5. On the relation between $\mathfrak{c}(G)$ and $\mathfrak{b}(G)$

The Perron root of the adjacency matrix of a graph G is also called the *index* of G and will be denoted henceforth by $\lambda_1(G)$.

The reason for $\mathfrak{c}(G) \geq \mathfrak{b}(G)$ is that it is possible that two non isomorphic connected induced subgraphs may have the same index. Consider, for example, the cycle C_4 with 4 vertices and the star S_5 with 5 vertices, both have index equal 2. They may be seen as induced subgraphs of the graph G composed by a vertex connecting one vertex of each C_4 and S_5.

We see that a necessary and sufficient condition for a graph G to have $\mathfrak{c}(G) = \mathfrak{b}(G)$ is that all non isomorphic connected subgraphs of G have different indices. Hence, we raise the following problem, which is interesting on its own for the spectral graph theory community, and also for estimating the maximum number of complementary eigenvalues of the adjacency matrix.

Question 2. *Characterize graphs G in which all induced connected subgraphs have different indices, i.e., graphs G such that $\mathfrak{c}(G) = \mathfrak{b}(G)$?*

We observe that the path P_n, the complete graph K_n, the star $S_n = K_{1,n-1}$ and the cycle C_n, all with n vertices, are examples of graphs G having exactly n non isomorphic connected induced subgraphs, each having different indices, meaning that $\mathfrak{c}(G) = \mathfrak{b}(G) = n$.

Question 2 asks to characterize graphs G for which $\mathfrak{b}(G) = \mathfrak{c}(G)$, which may be hard in general. Let us now turn to a problem of bounding these quantities. We are first going to study the number $\mathfrak{b}(G)$ of non isomorphic connected induced subgraphs of G. The following a priori result may be stated.

Theorem 5. *Let G be a connected graph G with n vertices. Then the number $\mathfrak{b}(G)$ of non isomorphic connected induced subgraphs of G satisfies*

$$n \leq \mathfrak{b}(G) < 2^n - 1. \tag{18}$$

Proof. In order to prove this result, let G be a connected graph with n vertices. Hence G provides the first induced graph. Now, we can always choose a vertex whose removal provides a connected induced subgraph with $n-1$ vertices. This process may be repeated until finding a graph of a single vertex. We have, therefore, exhibited n induced connected subgraphs of G and they are all non isomorphic since they have different number of vertices.

The upper bound follows because for each $k = 0, \ldots, n$ there are exactly $\binom{n}{k}$ ways of choosing k vertices. Hence, if all choices give connected and distinct subgraphs, then we would have 2^n such subgraphs. Now, we notice there is no connected subgraph with 0 vertices, there is only one subgraph with 1 vertex, there is a single connected subgraph with 2 vertices and hence we can improve the bound to at least $2^n - 1 + 1 - \binom{n}{1} + 1 - \binom{n}{2}$. Hence $\mathfrak{b}(G) < 2^n - 1$. $\square$

Let $\mathfrak{G}_n$ be set of all connected graphs on n vertices and let

$$\begin{aligned} \mathfrak{c}_n &= \max_{G \in \mathfrak{G}_n} \mathfrak{c}(G), \\ \mathfrak{b}_n &= \max_{G \in \mathfrak{G}_n} \mathfrak{b}(G). \end{aligned}$$

The lower bound $n \leq \mathfrak{b}_n$ given by Theorem 5 is indeed attained by the path P_n, by the star S_n, by the complete graph K_n and by the cycle C_n. It is worth noticing that for these graphs both $\mathfrak{b}(G)$ and $\mathfrak{c}(G)$ match the lower bound for $\mathfrak{b}_n$. From Theorem 4, we see that $\mathfrak{c}_n \leq \mathfrak{b}_n$ and since $\mathfrak{b}_n \geq n$, we ask whether there are graphs G for which the number of complementary eigenvalues is smaller than n.

Question 3. *Does there exist a connected graph G with n vertices for which $\mathfrak{c}(G) < n$?*

Clearly, the upper bound $\mathfrak{b}_n \leq 2^n - 1$ given by Theorem 5 can be improved if we take into account that many subgraphs are not connected or are isomorphic. Since our main interest here is in the number $\mathfrak{c}(G)$ of complementary eigenvalues, we leave as future research problem to find sharper upper bounds for $\mathfrak{b}_n$.

Due to the inequality (18), we may ask whether $\mathfrak{b}_n$ and $\mathfrak{c}_n$ are exponential in the number n. In the next sections, we study this topic. We will provide examples of family of graphs with n vertices for which the quantities $\mathfrak{b}_n$ and $\mathfrak{c}_n$ grow faster than any polynomial in n. To make it a more precise statement, we say that a quantity a_n grows faster than any polynomial in n if

$$\lim_{n \to \infty} \frac{a_n}{n^k} = \infty, \quad \text{for any } k > 1. \tag{19}$$

6. An example for $\mathfrak{b}_n$

It may be questioned whether the upper bound given by Theorem 5 can be improved so that $\mathfrak{b}(G)$ is a polynomial function of n. In this section we give an example of a graph whose number of connected non isomorphic induced subgraphs is larger than any polynomial. A *starlike* is a tree that has a unique vertex with maximum degree $r \geq 3$. We may picture a starlike with this unique vertex of degree $r > 2$ having r attached paths of sizes $m_1, m_2, \ldots, m_r$. It is convenient to denote such a starlike by $S(m_1, m_2, \ldots, m_r)$. In this section, we consider the starlike $S(r, m)$ having a vertex with $r > 2$ attached paths P_m all with $m > 0$ vertices each. Figure 2 gives an illustration.

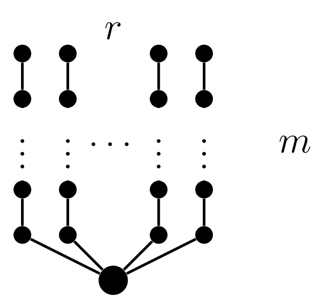

FIGURE 2. A starlike $S(r, m)$.

Lemma 2. *Let $S(r,m)$ with $m, r \geq 2$. The number of non isomorphic connected induced subgraphs of $S(r,m)$ is given by*

$$\binom{m+r}{r} - \binom{m}{2}.$$

Proof. In order to simplify the argument, we first notice that each non isomorphic connected induced subgraph may be seen as an r-tuple $(x_1, \ldots, x_r)$, with $0 \leq x_i \leq m$, and $x_i \geq x_{i+1}$, where each x_i represents the number of nodes of the respective path that appears in the subgraph. The restriction $x_i \geq x_{i+1}$ ensures that only non isomorphic subgraphs are counted, except in the case when $x_i = 0$ for $i > 2$. When only $x_1 \neq 0$ or $x_2 \neq 0$, then some subgraphs are multi counted. For example (2,2,0), (3,1,0) and (4,0,0) represent the same path P_5 as subgraphs of the starlike $S(3,3)$. We first count the total number of r-tuples with the restriction $x_i \geq x_{i+1}$ and then account for the repetitions.

Let $k \in \{1, \ldots, r\}$ be the number of distinct values that appear in the r-tuple. We see there are $\binom{m+1}{k}$ ways of choosing k distinct values from $\{0, 1, \ldots, m\}$. For each of these choices, we need to choose the configuration, that is the number of each individual value. For example, for $r = 4, k = 2$, if we choose the values 1 and 3, we may have the 4-uples (3,3,3,1), (3,3,1,1) or (3,1,1,1). In general the number of configurations is equal to the number of positive integral solutions of the equation $y_1 + \cdots + y_k = r$, which is known to be equal to $\binom{r-1}{k-1}$. Hence the total count is

$$k_r = \sum_{k=1}^{r} \binom{m+1}{k}\binom{r-1}{k-1}.$$

Let us now count the number of paths given by the $(x_1, x_2, 0, \cdots, 0)$, with $x_1 \geq x_2 > 0$. We can use the same reasoning above to arrive at $k_2 = \sum_{k=1}^{2} \binom{m}{k}\binom{r-1}{k-1} = \binom{m+1}{2}$. Notice that $k_r - k_2$ counts all unlabelled subgraphs, except the paths with more than $m+1$ vertices. Now these paths may be seen as the tuples of the form $(m, l, 0, \cdots, 0)$ for $l = 1, \ldots, m$, which gives m additional paths. Hence, the number of non isomorphic connected induced subgraphs of $S(r,m)$ is given by $k_r - k_2 + m$, that is

$$\sum_{k=1}^{r} \binom{r-1}{k-1}\binom{m+1}{k} + m - \binom{m+1}{2}. \tag{20}$$

Using Vandermonde's convolution formula $\binom{l+s}{n} = \sum_k \binom{r}{k}\binom{s}{n-k}$ (see, for example [15, p.170]), we see that $k_r = \sum_{k=1}^{r} \binom{m+1}{k}\binom{r-1}{k-1} = \sum_{k=1}^{r} \binom{m+1}{k}\binom{r-1}{r-k} = \binom{m+r}{r}$. Since $m - \binom{m+1}{2} = -\binom{m}{2}$, the result follows. $\square$

By applying Stirling's formula (see, for example, [15, p.519]), one may conclude that

$$\binom{m+r}{m} \approx \sqrt{\frac{1}{2\pi}\left(\frac{1}{m} + \frac{1}{r}\right)}\left(1 + \frac{r}{m}\right)^m \left(1 + \frac{m}{r}\right)^r.$$

If we take $m = r$, we arrive at the following result.

Corollary 1. *The number $\mathfrak{b}(G)$ of non isomorphic connected induced subgraphs of the starlike $G = S(m,m), m > 1$ with $n = m^2 + 1$ vertices is asymptotically given by*

$$\frac{2^{2\sqrt{n}}}{\sqrt{\pi\sqrt{n}}}.$$

Theorem 6. *For all natural $n > 3$, there exists a graph G_n having n vertices whose number $\mathfrak{b}(G_n)$ of non isomorphic connected induced subgraphs grows faster than any polynomial in n.*

Proof. For a given natural n, we can always place it between two consecutive squares: $r^2 < n \leq (r+1)^2$. Hence there exists an integer $1 \leq l \leq 2r+1$ such that $n = r^2 + l$. We now construct a starlike $S(r, m)$, with m satisfying $\begin{cases} m = r & \text{if} \quad 1 \leq l \leq r \\ m = r+1 & \text{if} \quad r+1 \leq l \leq 2r+1. \end{cases}$

The number of vertices left out is $n-(mr+1) = \begin{cases} l-1 & \text{if} \quad 1 \leq l \leq r \\ l-r-1 & \text{if} \quad r+1 \leq l \leq 2r+1. \end{cases}$

In any case, we have $0 \leq n - (mr + 1) \leq r$. Then we can place one vertex at the end of each path, leaving all paths with at most $m + 1$ vertices. The final starlike tree G_n with n vertices has a unique vertex of degree r, and the r paths have either m or $m+1$ vertices. The number of induced non isomorphic connected subgraphs of G_n is larger than $S(r, m)$ and this counting may be done using the technique of Lemma 5. Since $r \approx m \approx \sqrt{n}$, we may apply Stirling formula as in Corollary 1, and it is clear that $\lim_{n\to\infty} \frac{\mathfrak{c}(G_n)}{n^k} = \infty$ for any $k > 1$, hence we obtain the result. $\square$

Example 2. *For $n = 14$, the construction above gives an initial $S(r, m)$ with $(r, m) = (3, 4)$, since $3^2 < n = 14 \leq (3+1)^2$ and, hence $n = r^2 + l = 3^2 + 5$. There is one vertex left out to be placed at the end of a path. The final G_{14} is a starlike with a central vertex of degree 3 attached to 2 paths P_4 and 1 path P_5.*

7. An example for $\mathfrak{c}_n$

In this section, we show that the number of complementary eigenvalues of graphs may also grow faster than polynomially with the number of vertices. We are going to look at starlike trees as in the previous section. We should proceed with care because non isomorphic starlike trees may have the same index, therefore leading to the same complementary eigenvalue. For example, the starlike $S(3, 2)$ with 7 vertices and the star $S(4, 1)$ with 5 vertices both have index 2. They may be seen as induced subgraphs of the same larger starlike. For an account about the index of starlike trees, we refer to [22], where conditions are given for the indices of starlikes to be integers.

Consider now a general starlike tree $S(m_1, m_2, \ldots, m_r)$ having $r > 2$ paths P_{m_i} attached to a vertex. Let $\mathcal{S}_n$ be the set of all starlike trees with $n > 3$ vertices. The following is a recent result that is essential to our example (see [8])./

Lemma 3. *Any two non isomorphic starlike trees in $\mathcal{S}_n$ have different indices.*

Lemma 4 (Cardinality of $\mathcal{S}_n$)**.** *Let $n > 3$ be an integer. The number of starlike trees of $\mathcal{S}_n$ is asymptotically given by*

$$\frac{1}{4n\sqrt{3}} \exp \pi \sqrt{\frac{2n}{3}}.$$

Proof. We first set $m_1 \geq m_2 \geq \cdots \geq m_r > 0$ to guarantee that the starlikes are non isomorphic. Then it is easy to see that counting the number of starlike trees $S(m_1, \ldots, m_r)$ of $\mathcal{S}_n$ and $3 \leq r \leq n - 1$ paths, is equivalent to counting the number of different configurations of $(m_1, \ldots, m_r)$ such that $m_1 + m_2 + \cdots + m_r = n - 1$. But this is exactly the number of partitions of $n - 1$ having at least three parts. If we let

$p(k)$ be the number partitions of k (with no restriction) then our number accounts to $p(n-1) - \left(\lfloor \frac{n-1}{2} + 1 \rfloor\right)$. The famous Hardy-Ramanujan approximation, states that

$$p(k) \quad \sim \quad \frac{1}{4k\sqrt{3}} \exp \pi \sqrt{\frac{2k}{3}}.$$

Since replacing n by $n-1$ and subtracting the linear term does not change the asymptotic behavior, the result follows. □

Corollary 2. *Let G be the starlike $G = S(m,m), m > 2$ with $n = m^2 + 1$ vertices. The number $\mathfrak{c}(G)$ of distinct complementary eigenvalues of G grows asymptotically at least as*

$$\frac{\exp \pi \sqrt{2\sqrt{n}/3}}{4\sqrt{3n}}.$$

Proof. Notice that any starlike having m vertices will have at most $m - 1 = \sqrt{n}$ paths and the largest path can not have more than $m - 1 = \sqrt{n}$ vertices, meaning that it is a connected subgraph of $G = S(m,m)$. By Lemma 3 they are all distinct and, therefore, contribute with a complementary eigenvalue of G. Now the asymptotic value follows from Lemma 4. □

Theorem 7. *For all $n > 3$, there exists a graph G_n with n vertices whose number $\mathfrak{c}(G_n)$ of distinct complementary eigenvalues grows faster than any polynomial in n.*

Proof. For a given $n > 3$, we construct the same starlike G_n from $S(r,m)$ of Theorem 6 with $r^2 < n + l \leq (r+1)^2$, $0 \leq l \leq 2r + 1$ and $m = r$ or $r + 1$. Then, using Corollary 2, as $m \approx \sqrt{n}$, the number $\mathfrak{c}(G_n)$ of distinct complementary eigenvalues of G_n is asymptotically faster than any polynomial in n. □

8. Concluding remarks

We studied the EiCP whose matrix A belongs to the class $S(G)$ associated to a connected graph $G(V,E)$ defined by (6). We showed that any matrix $A \in S(G)$ has a unique complementary eigenvalue if all its non diagonal elements are non positive. As a consequence, zero is the only complementary eigenvalue of the Laplacian and the normalized Laplacian matrices of G. We also studied the number $\mathfrak{c}(G)$ of complementary eigenvalues of the adjacency matrix of an undirected graph G. We showed that $\mathfrak{c}(G)$ is bounded above by the number of induced connected subgraphs of G. Furthermore, we were able to show that the maximum number of complementary eigenvalues of adjacency matrices of graphs is asymptotically faster than any polynomial in the number of nodes. Some questions are included that should be considered open problems. The investigation of classes of graphs for which the maximum number of complementary eigenvalues is polynomial in the number of nodes is an interesting area of future research. The design of special purpose techniques for computing complementary eigenvalues of adjacency matrices of these graphs is certainly another topic of interest in our future research.

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