Algorithms for Linear Programming with Linear Complementarity Constraints

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Abstract

Linear programming with linear complementarity constraints (LPLCC) is an area of active research in Optimization, due to its many applications, algorithms, and theoretical existence results. In this paper, a number of formulations for important nonconvex optimization problems are first reviewed. The most relevant algorithms for computing a complementary feasible solution, a stationary point, and a global minimum for the LPLCC are also surveyed, together with some comments about their efficiency and efficacy in practice.

Key words: Complementarity Problems, Global Optimization, Nonlinear Programming, Mathematical Programming with Linear Complementarity Constraints.

Mathematics Subject Classification: 90C26, 90C30, 90C33

1 Introduction

A mathematical program with linear complementarity constraints (MPLCC) [15, 44, 51, 53] consists of minimizing a continuously differentiable function on a set defined by a general linear complementarity problem (GLCP). Therefore the problem can be stated as follows:

$$\begin{array}{rl} Minimize & f(x,y) \\ subject\ to & Ew = q + Mx + Ny \\ & x_i \geq 0,\, w_i \geq 0,\, i \in I \\ & x_i w_i = 0,\, i \in I \\ & w_j = 0,\, j \in \{1,\ldots,n\} \backslash I \\ & y \in K_y, \end{array} \tag{1}$$

where $I \subseteq \{1,\ldots,n\}$, $E \in \mathbb{R}^{p\times n}$, $M \in \mathbb{R}^{p\times n}$, $N \in \mathbb{R}^{p\times m}$, and K_y is a polyhedron in $\mathbb{R}^m$. Since the pioneering work of Ibaraki [29] and Jeroslow [30] in the 1970s, many theoretical results, algorithms, and applications of the MPLCC have been reported (see [44, 51, 53] for important monographs on the MPLCC). The MPLCC is called a linear program with linear complementarity constraints (LPLCC) if the function f is linear, that is, it takes the form

$$f(x,y) = c^T x + d^T y \tag{2}$$

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where $c \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$. The linear bilevel programming problem has been reformulated as an LPLCC [6, 11, 14] and this formulation was used in the 1990s in the design of algorithms for finding a global minimum for the former problem [7, 25, 35]. Other formulations of nonconvex programs as LPLCC and many applications of this last problem have been discussed in the past several years [44, 51, 53]. Recently, a great interest has reemerged on the design of new efficient algorithms for finding stationary points and global minima for the LPLCC [5, 10, 17, 26, 27, 37, 38, 61]. These methods either solve the LPLCC directly or find a solution to an equivalent nonconvex program. The main objective of this paper is to survey the most important techniques for the LPLCC.

The GLCP is a nonconvex and NP-hard problem [39] that has been studied in the past several years [39, 44, 51, 64]. A special case of the GLCP is the so called mixed linear complementarity problem (mixed LCP), where the y-variables do not exist in its definition. Several direct and iterative methods have been discussed for processing the mixed LCP [12, 49]. These techniques can be highly efficient for special classes of matrices, namely, when the matrix M is positive semi-definite (PSD) and E is the identity matrix or belong to related classes of matrices [12, 49]. The extensions of these methods to the GLCP in these special cases are straightforward [18, 39, 64]. Absolute value and DC programming [41, 46] have also been recommended for the solution of the LCP and can be extended to deal with the GLCP without any major modification. In general, the mixed LCP and the GLCP can only be processed by an enumerative method. An efficient algorithm of this type was introduced in [1] and was subsequently improved in [33, 37]. The algorithm searches a solution to the GLCP by exploiting a binary tree that is constructed based on the dichotomy that $x_i = 0$ or $w_i = 0$, as associated with the complementary variables. A set of heuristic rules and a local quadratic solver, such as MINOS [47], were incorporated in the algorithm to speed up the search for a solution.

Due to the nonnegativity requirement for the variables x_i and w_i, $i \in I$, the $|I|$ constraints $x_i w_i = 0$, $i \in I$ can be replaced by a complementarity constraint

$$\sum_{i \in I} x_i w_i = 0. \tag{3}$$

Therefore the LPLCC can be seen as a nonlinear programming problem (NLP) with linear constraints and a nonlinear equality restriction. A number of stationary concepts have been associated with the LPLCC [17, 44, 51, 52, 53, 55, 62, 63]. Among them, strongly stationary and B-stationary points are particularity noteworthy [17, 53]. Many algorithms have been developed in the past several years for finding a stationary point for the MPLCC and LPLCC [2, 3, 9, 15, 16, 17, 19, 20, 21, 22, 28, 31, 32, 38, 43, 44, 51, 53, 56, 57]. As stated in [17], the complementary active-set algorithm (CASET) developed in [56] and subsequently improved in [17, 38] is the most recommended approach for finding a strongly stationary or a B-stationary point for an LPLCC. This algorithm exploits an active-set methodology that only employs solutions of the GLCP.

A sequential linear complementarity (SLCP) algorithm was introduced in [35] for computing a global minimum of a linear bilevel program by exploiting its LPLCC formulation and has subsequently been applied to other optimization problems and to the general LPLCC [34, 36]. The algorithm achieves global convergence to an approximate global minimum of the LPLCC, but in practice, is able to compute a true global minimum. This procedure has essentially two phases, namely, a parametric enumerative procedure (PAREN) that is able to compute solutions of the GLCP with strictly decreasing values for the objective function, and the complementarity active-set (CASET) algorithm, which is applied a finite number of times starting from these solutions of the GLCP found by the PAREN method to derive stationary points [17, 38, 56] of the LPLCC. Branch-and-bound algorithms [4, 5, 7, 10, 13, 25, 37, 61] have also been proposed for finding a global minimum to the LPLCC. Similar to the enumerative method, this type of algorithm explores a binary tree that is generated based on the dichotomy that $x_i = 0$ or $w_i = 0$, as per the complementarity constraints $x_i w_i = 0, i \in I$. The LPLCC can also be reduced to a $0-1$ mixed-integer program (MIP) [24, 26, 27, 45, 59] and solved by special-purpose techniques that deal with such problems.

The remainder of this paper is organized as follows. In Section 2, we recall some formulations of optimization problems as LPLCCs. The solution of the GLCP is discussed in Section 3. The computation of strongly stationary and B-stationary points for the LPLCC is addressed in Section 4. The sequential complementarity algorithm and the branch-and-bound methods are described in Sections 5 and 6. The solution of the LPLCC via MIPs is studied in Section 7. Some concluding remarks are presented in the final section of the paper.

2 Formulations of nonconvex programs as LPLCCs

The definition of a bilevel programming problem (BPP) [6, 11, 14] contains a hierarchy between two optimization problems, such that the constraints of the upper (first) level problem are defined as part of a parametric optimization problem, called the lower or second level problem. The BPP can be stated as follows:

$$\begin{array}{ll} Minimize & f(x,y) \\ subject\ to & y \in K_y \\ & G_1(x,y) \geq 0 \\ & G_2(x,y) = 0 \\ & x \in arg\min\{g(x,y) : x \in K_x,\ H_1(x,y) \geq 0,\ H_2(x,y) = 0\}, \end{array}$$

where K_x and K_y are polyhedra in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, $f : \mathbb{R}^{n+m} \to \mathbb{R}$ and $g : \mathbb{R}^{n+m} \to \mathbb{R}$ are real functions, and G_i and $H_i,\ i = 1, 2$ are vector real functions, such that $G_i : \mathbb{R}^{n+m} \to \mathbb{R}^{l_i}$ and $H_i : \mathbb{R}^{n+m} \to \mathbb{R}^{p_i}$, n, $m \in \mathbb{N}$ and $l_i, p_i \in \mathbb{N} \cup \{0\}, i = 1, 2$. It follows from its definition that the functions G_i, H_i, g and f have an important impact on the difficulty of the BPP and the choice of the techniques to deal with the problem. In this paper we assume that G_i and H_i are all linear functions, f is linear, and g is quadratic. Hence the BPP can be defined as follows:

$$\begin{array}{ll} Minimize & g^T x + h^T y \\ subject\ to & Cx + Dy = r \\ & y \in K_y, \end{array}$$

where x is the optimal solution of the parametric quadratic convex program

$$\begin{array}{ll} Minimize & c^T x + y^T Rx + \frac{1}{2} x^T Qx \\ subject\ to & Ax + By = b \\ & x \geq 0. \end{array}$$

Replacing the second level quadratic problem by its *Karush-Kuhn-Tucker* conditions [8], the BPP reduces to the following LPLCC:

$$\begin{array}{ll} Minimize & g^T x + h^T y \\ subject\ to & c + Qx + R^T y = A^T u + w \\ & Ax + By = b \\ & Cx + Dy = r \\ & x \geq 0,\ w \geq 0,\ y \in K_y \\ & x^T w = 0. \end{array} \tag{4}$$

If no first level constraints $Cx + Dy = r$ exist, then the GLCP of the last LPLCC takes the form

$$\begin{bmatrix} I_n \\ 0 \end{bmatrix} w = \begin{bmatrix} c \\ -b \end{bmatrix} + \begin{bmatrix} Q & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} R^T \\ B \end{bmatrix} y$$
$$x \geq 0,\ w \geq 0,\ y \in K_y$$
$$x^T w = 0,$$

where I_n is the identity matrix of order n. An important fact of this LPLCC is that the matrix

$$\begin{bmatrix} Q & -A^T \\ A & 0 \end{bmatrix}$$

involving the complementary variables is PSD, as $Q \in$ PSD [12]. This issue is discussed later in this paper.

The bilinear programming problem (BLP) has also been extensively investigated due to its large number of applications [4, 40, 58]. It consists of minimizing a bilinear function in the variables x_i and y_i on a convex set defined by linear constraints. Therefore, it takes the form

$$\begin{array}{ll} Minimize & f(x, y) = c^T x + d^T y + x^T H y \\ subject\ to & Ax + By = b \\ & x \in K_x,\ \ y \in K_y, \end{array} \tag{5}$$

where $K_x \subseteq \mathbb{R}^n$ and $K_y \subseteq \mathbb{R}^m$ are polyhedra in the x- and y-variables, respectively. If

$$K_x = \{x \in \mathbb{R}^n : Cx = g, x \geq 0\},$$

then BLP can be stated as follows:

$$\begin{array}{c} Minimize \\ y \in K_y \end{array} \quad d^T y + \min_x \left\{ (c + Hy)^T x :\ Ax + By = b,\ Cx = g,\ x \geq 0 \right\}. \tag{6}$$

The dual program of the inner program above is given by

$$\begin{array}{ll} Maximize & (b - By)^T u + g^T v \\ subject\ to & A^T u + C^T v \leq c + Hy. \end{array} \tag{7}$$

By introducing the slack variables w_i related to the inequality constraints of the dual program, and applying the complementary slackness theorem [48], $(\bar{x}, \bar{y})$ is an optimal solution of BLP (5) if and only if $(\bar{x}, \bar{y})$ is a global minimum of the MPLCC

$$\begin{array}{ll} Minimize & d^T y + g^T v + b^T u - u^T B y \\ subject\ to & \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix} w = \begin{bmatrix} c \\ -b \\ -g \end{bmatrix} + \begin{bmatrix} 0 & -A^T & -C^T \\ A & 0 & 0 \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \\ v \end{bmatrix} + \begin{bmatrix} H \\ B \\ 0 \end{bmatrix} y \\ & x \geq 0,\ w \geq 0,\ y \in K_y \\ & x^T w = 0. \end{array}$$

Note that the matrix

$$\begin{bmatrix} 0 & -A^T & -C^T \\ A & 0 & 0 \\ C & 0 & 0 \end{bmatrix}$$

is skew-symmetric and thus PSD [12]. A BLP is called *disjoint* if there are no constraints involving both x- and y-variables. So a disjoint bilinear program is equivalent to an LPLCC of the form

$$\begin{array}{ll} Minimize & d^Ty + g^Tv \\ subject\ to & \left[\begin{array}{c} I_n \\ 0 \end{array}\right] w = \left[\begin{array}{c} c \\ -g \end{array}\right] + \left[\begin{array}{cc} 0 & -C^T \\ C & 0 \end{array}\right] \left[\begin{array}{c} x \\ v \end{array}\right] + \left[\begin{array}{c} H \\ 0 \end{array}\right] y \\ & x \geq 0,\ w \geq 0,\ y \in K_y \\ & x^Tw = 0. \end{array}$$

As before, the matrix involving the complementarity variables is PSD.

As stated in [10, 26, 61], a nonconvex quadratic program (QP) can also be solved by exploiting an equivalent LPLCC formulation. Consider the following QP:

$$\begin{array}{ll} Minimize & c^Tx + \frac{1}{2}x^THx \\ subject\ to & Ax = b \\ & x \geq 0, \end{array} \tag{8}$$

where $H \in \mathbb{R}^{n\times n}$ is symmetric and not PSD, $A \in \mathbb{R}^{m\times n}$ with $m < n$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Let u and w represent the dual variables associated with the primal constraints. If $\bar{x}$ is an optimal solution for QP then there are vectors $\bar{u}$ and $\bar{w}$ such that $(\bar{x}, \bar{u}, \bar{w})$ is a solution of the GLCP

$$\begin{array}{c} \left[\begin{array}{c} I_n \\ 0 \end{array}\right] w = \left[\begin{array}{c} c \\ -b \end{array}\right] + \left[\begin{array}{cc} H & -A^T \\ A & 0 \end{array}\right] \left[\begin{array}{c} x \\ u \end{array}\right] \\ x \geq 0,\ w \geq 0 \\ x^Tw = 0. \end{array} \tag{9}$$

Furthermore, for each solution of the GLCP the QP objective function is linear in the variables x and u, because

$$c^Tx + \frac{1}{2}x^THx = \frac{1}{2}c^Tx + \frac{1}{2}x^T(c + Hx) = \frac{1}{2}(c^Tx + b^Tu).$$

Hence, any nonconvex QP with an optimal solution is equivalent to an LPLCC. Note that the matrix associated with the complementary variables is not PSD.

Absolute value programming (AVP) has been studied in [46] and is also related to mathematical programming problems with linear complementarity constraints. Consider the following AVP problem:

$$\begin{array}{ll} Minimize & c^Tx + d^T|x| \\ subject\ to & Ax + B|x| = b, \end{array} \tag{10}$$

where $c, d, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A, B \in \mathbb{R}^{m\times n}$, and where $|x| \in \mathbb{R}^n$ denotes the vector whose i[th] component is the absolute value of x_i. For each i, we can write

$$x_i = u_i - v_i, \quad u_i \geq 0, v_i \geq 0, \quad u_iv_i = 0.$$

Then

$$|x_i| = u_i + v_i,$$

and the AVP is equivalent to the following LPLCC:

$$\begin{array}{ll} Minimize & (c + d)^Tu + (d - c)^Tv \\ subject\ to & (A + B)u + (B - A)v = b \\ & u \geq 0,\ v \geq 0 \\ & u^Tv = 0. \end{array}$$

As discussed in [48], there are some special cases where the complementarity constraint is redundant and the AVP reduces to a simple linear program.

3 Solution of the general linear complementarity problem

Consider again the GLCP introduced in Section 1:

$$Ew = q + Mx + Ny \tag{11}$$
$$x_i \geq 0, w_i \geq 0,\ i \in I \tag{12}$$
$$w_j = 0,\ j \in \{1, \ldots, n\} \backslash I \tag{13}$$
$$y \in K_y \tag{14}$$
$$\sum_{i \in I} x_i w_i = 0, \tag{15}$$

where $I \subseteq \{1, \ldots, n\},\ w, x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $q \in \mathbb{R}^p$, $E, M \in \mathbb{R}^{p \times n}$, $N \in \mathbb{R}^{p \times m}$, and where K_y is a polyhedron defined by

$$K_y = \{y \in \mathbb{R}^m \colon Ay = b, y \geq 0\}, \tag{16}$$

with $A \in \mathbb{R}^{t \times m}$ and $b \in \mathbb{R}^t$. It follows from its definition that the GLCP contains a set of linear constraints (11) - (14) and (16) and the complementarity condition (15). As in linear programming, $(\bar{x}, \bar{y}, \bar{w})$ is a *feasible solution* for the GLCP if it satisfies the linear constraints. Furthermore, the GLCP is *feasible* if there exists at least a feasible solution and *infeasible*, otherwise. A solution $(\bar{x}, \bar{y}, \bar{w})$ is called *complementary* if satisfies the constraints $x_i w_i = 0$ for all $i \in I$. Furthermore, it is called *nondegenerate* if $\bar{x}_i + \bar{w}_i \neq 0$ for all $i \in I$. Otherwise it is said to be *degenerate*.

It follows from the definitions above that a solution of the GLCP must be feasible and complementary. Furthermore a feasible GLCP may have a solution or not. This can be checked by considering the following nonconvex quadratic program

$$\begin{array}{lll} (QP) & Minimize & \sum_{i \in I} x_i w_i \\ & subject\ to & (11) - (14). \end{array} \tag{17}$$

Since the objective function is bounded from below on the constraint set of the QP, this problem has at least a global minimum $(\bar{x}, \bar{y}, \bar{w})$ [26] and two cases may occur:

(i) $\sum_{i \in I} \bar{x}_i \bar{w}_i = 0$, so that $(\bar{x}, \bar{y}, \bar{w})$ is a solution of the GLCP.

(ii) $\sum_{i \in I} \bar{x}_i \bar{w}_i > 0$, and so the GLCP has no solution, that is, it is *unsolvable*.

So a GLCP is either solvable or unsolvable (feasible or infeasible). Finding whether a GLCP is feasible is easy, as this reduces to solving a linear program. However, checking whether a feasible GLCP is solvable is an NP-hard problem [39], because this is equivalent to finding a global minimum of a nonconvex quadratic program.

Despite being NP-hard, the GLCP can be solved relatively easily for some special cases. In the previous section, we discussed a number of formulations of nonconvex programs that lead to a GLCP of the form

$$\begin{array}{c} \begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} q \\ -d \end{bmatrix} + \begin{bmatrix} H & -C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} R \\ S \end{bmatrix} y \\ Ay = b,\ y \geq 0 \\ x \geq 0, w \geq 0 \\ x^T w = 0. \end{array} \tag{18}$$

Consider the following associated QP:

$$\begin{array}{ll} Minimize & q^T x + \frac{1}{2} x^T (H + H^T) x - x^T C^T u + x^T R y \\ subject\ to & Hx - C^T u + Ry \geq -q \\ & Cx + Sy = d \\ & Ay = b \\ & x \geq 0, y \geq 0. \end{array} \tag{19}$$

By exploiting the special structure of the QP (19) and using a proof similar to the one presented in [39], it is possible to show that either the GLCP (18) is infeasible or solvable and any stationary (KKT) point of the QP (19) is a solution of this GLCP. Therefore the GLCP associated with LPLCC formulations of bilevel and bilinear programs introduced in the previous section can be solved by computing a stationary (KKT) point of the nonconvex quadratic program (19). This can be done by using an active-set method [23, 50], such as MINOS [47], or an interior-point algorithm [18, 64]. On the other hand, the GLCP (9) associated with the quadratic program (8) can be solved by finding a stationary point for this quadratic program using these techniques [50].

DC (difference of convex function) programming [42] has become quite popular in recent years for dealing with nonconvex programming problems and has been proposed in [41] for finding a solution of an LCP (GLCP with $I = \{1, \ldots, n\}$ and $m = 0$). The application of this approach to the GLCP is straightforward. Consider the formulation of the GLCP as the nonconvex quadratic program (17) and let $z = (x_I, x_J, w_I, w_J, y)^T \in \mathbb{R}^{2n+m}$ where $J = \{1, \ldots, n\} \backslash I$. Furthermore, let H be the symmetric matrix of order $(2n + m)$ whose columns $H_{.j}$ are defined as follows:

$$H_{.j} = \begin{cases} e^{n+j} & if\ \ j \in \{1, \ldots, |I|\} \\ e^{j-n} & if\ \ j \in \{n+1, \ldots, n + |I|\} \\ 0 & otherwise, \end{cases}$$

where $e^l \in \mathbb{R}^{2n+m}$ is the lth column of the identity matrix, and $|I|$ represents the number of elements of the set I. Because all the eigenvalues of H belong to the interval $[-1, 1]$, then

$$H + \rho I_{2n+m}$$

is a positive definite matrix for each $\rho > 1$, where I_{2n+m} is the identity matrix of order $(2n + m)$. Hence, there are infinite DC decompositions of the objective function of the quadratic program (17) as

$$\begin{aligned} \sum_{i \in I} x_i w_i &= z^T (H + \rho I_{2n+m}) z - \rho ||z||_2^2 \\ &= g_\rho(z) - h_\rho(z), \end{aligned}$$

where $||z||_2$ denotes the Euclidean norm of the vector z.

By using one of these decompositions for a fixed $\rho > 1$, a DC Algorithm (DCA) has been introduced in [41], which can be used without any modification to find a stationary point of the nonconvex quadratic program (17). To describe an iteration of this algorithm, let $\bar{z}$ be a current feasible solution of the program (17), that is, a feasible solution to the GLCP. The gradient $\nabla h_\rho(\bar{z})$ of $h_\rho(z)$ at $\bar{z}$ is first computed by

$$\bar{u} = \nabla h_\rho(\bar{z}) = 2\rho \bar{z}.$$

Then, the following strictly convex quadratic program is considered:

$$\begin{array}{ll} Minimize & z^T (H + \rho I_{2n+m}) z - \bar{u}^T z \\ subject\ to & (11) - (14). \end{array}$$

Because the GLCP is feasible, this quadratic program is also feasible and has a unique optimal global solution. Now, either $\bar{z} \simeq \tilde{z}$ and $\bar{z}$ is a stationary point of (17), or a new iteration has to be performed with $\tilde{z}$ as the new point.

As discussed in [41], DCA achieves global convergence to a stationary point of the quadratic program (17). Hence DCA is able to find a solution of a GLCP of the form (18) when H is a PSD matrix, but there is no theoretical guarantee that the algorithm is successful in general. Computational experiments reported in [41] show that the efficiency and efficacy for finding a solution of the GLCP (that is a global optimal solution of the quadratic program (17)) depends strongly on the initial point and the constant ρ used in the DC decomposition. The same conclusion should hold for the GLCP, which means that DCA can solve the GLCP efficiently in many cases but not always.

Absolute value programming (AVP) has also been shown to be an interesting approach for solving the LCP [46] and can be useful for solving the GLCP. Consider the GLCP where $I = \{1, \ldots, n\}$, $E = I_n$ is the identity matrix of order n and K_y is given by (16). As discussed in [46], by scaling the square matrix M if necessary, $I_n - M$ is nonsingular and the GLCP can be reduced to the following system

$$\begin{aligned}
(I_n + M)(I_n - M)^{-1}z - |z| &= -((I_n + M)(I_n - M)^{-1} + I_n)(q + Ny) && (20)\\
x &= (I_n - M)^{-1}(z + q + Ny) && (21)\\
w &= q + Mx + Ny && (22)\\
Ay &= b && (23)\\
y &\geq 0. && (24)
\end{aligned}$$

It immediately follows from (21) and (22) that $w = x - z$ in any solution of the system. According to [46], let $s \in \mathbb{R}^n$ and $t \in \mathbb{R}^n$ be two additional vectors and consider the following concave program (CP):

$$\begin{array}{ll}
Minimize & \rho e^T(t - |z|) + e^T s = f(s,t,x,y,z) \\
subject\ to & (I_n - M)x = z + q + Ny \\
& -s \leq 2x - z - t \leq s \\
& -t \leq z \leq t \\
& Ay = b \\
& y \geq 0,
\end{array} \qquad (25)$$

where ρ is a positive real number (may be equal to one) and $e \in \mathbb{R}^n$ is a vector of ones. Then, it is easy to show that $(\bar{s}, \bar{t}, \bar{x}, \bar{y}, \bar{z})$ is a global minimum of CP (25) with $f(\bar{s}, \bar{t}, \bar{x}, \bar{y}, \bar{z}) = 0$ if and only if $(\bar{x}, \bar{y}, \bar{w} = \bar{x} - \bar{z})$ is a solution of the GLCP.

It is interesting to note that the CP (25) is a DC program and can be solved by a DC algorithm [42]. Alternatively, a sequential linear programming (SLP) algorithm introduced in [46] can be applied to find a stationary point of CP (25). As stated before, there is no theoretical guarantee that the method finds a global minimum of the CP. However, numerical experiments reported in [46] indicate that the algorithm is, in general, able to terminate successfully with a solution to the LCP. Hence, this approach appears to be interesting to exploit in the future for solving the GLCP in practice.

Since the GLCP is an NP-hard problem, an enumerative algorithm is required in general to solve it. This method [1, 33, 37] exploits a binary tree that is constructed based on the dichotomy that $x_i = 0$ or $w_i = 0$ for each complementary pair of variables.

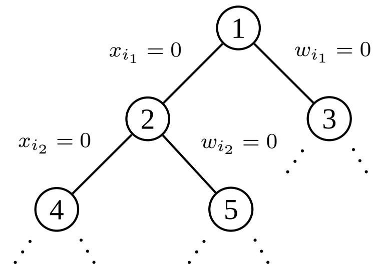

Figure 1: Branching on enumerative method.

In order to accelerate the search for a solution of the GLCP, each node of the tree in the algorithm attempts to compute a stationary point of the following quadratic program:

$$\begin{array}{ll} Minimize & \sum\limits_{i \in I} x_i w_i \\ subject\ to & Ew = q + Mx + Ny \\ & x_i = 0,\ i \in F_x \\ & w_j = 0,\ j \in F_w \\ & w_i \geq 0, x_i \geq 0, i \in I \\ & w_j = 0,\ j \in \{1, \ldots, n\} \backslash I \\ & y \in K_y, \end{array} \tag{26}$$

with $F_x \subseteq I$ and $F_w \subseteq I$ being the sets defined by the fixed x- and w-variables in the path of the tree from the root to this node. Now, either the program (26) is infeasible and the node is pruned, or a stationary point $(\bar{x}, \bar{y}, \bar{w})$ can be computed by a local optimization algorithm. Based on this, two cases may occur:

(i) $\sum\limits_{i \in I} \bar{x}_i \bar{w}_i = 0$, whence a solution of the GLCP has been attained and the algorithm stops.

(ii) $\sum\limits_{i \in I} \bar{x}_i \bar{w}_i > 0$, whence two nodes have to be generated for a pair of complementary variables (x_i, w_i) such that $\bar{x}_i > 0$ and $\bar{w}_i > 0$.

A good implementation of an enumerative algorithm requires some heuristic rules for selecting the pair of complementary variables and for choosing a node from the set of open nodes of the tree to be investigated next. These issues are discussed in [33]. Furthermore, the stationary points of the quadratic program can be computed by using an active-set method [23, 50], such as MINOS [47].

The foregoing brief description of the algorithm immediately leads to the conclusion that it essentially looks for a stationary point of the complementarity gap function having a zero objective function value in order to find a solution to the GLCP. If such a solution does not exist but the linear constraints are feasible, then an extensive search needs to be typically performed in the tree before the algorithm terminates. This feature makes the algorithm particulary suitable for finding a solution to the GLCP but almost impractical when dealing with a feasible GLCP that has no solution.

4 Finding a stationary point for the LPLCC

The MPLCC and LPLCC can be seen as nonlinear programming problems (NLPs) with linear constraints and a nonlinear equality (3). The special structure of this last constraint prevents the wellknown Mangasarian-Fromovitz constraint qualification [50] to hold. This fact has prompted an extensive research effort on new definitions of stationarity and constraint qualifications for the MPLCC [17, 44, 51, 52, 53, 55, 62, 63]. The concepts of B-stationary and strongly stationary points for the MPLCC (and LPLCC) play an important role on the solution of the LPLCC. In order to define such points, consider the MPLCC

$$\begin{array}{ll} Minimize & f(x,y) \\ subject\ to & (11)-(15), \end{array} \tag{27}$$

where K_y is the polyhedron defined by (16), and where $f\colon \mathbb{R}^{n+m} \to \mathbb{R}$ is a continuous differentiable function on an open set containing the constraint set of the MPLCC, that is, the solution set of the GLCP. Let $(\bar{x}, \bar{y}, \bar{w})$ be a solution of the GLCP and consider the following sets:

$$\begin{aligned} I_x &= \{i \in I\colon \bar{x}_i = 0\} \\ I_w &= \{i \in I\colon \bar{w}_i = 0\}. \end{aligned}$$

Then $(\bar{x}, \bar{y}, \bar{w})$ is a *B-stationary point* [17] if and only if it is a stationary (KKT) point of all the nonlinear programs NLP(L), $L \subseteq I_x \cap I_w$, of the following form:

$$\begin{array}{ll} Minimize & f(x,y) \\ subject\ to & Ew = q + Mx + Ny \\ & Ay = b \\ & y \geq 0 \\ & x_i = 0, w_i \geq 0, \quad i \in I \backslash (I_x \cap I_w) \text{ and } \bar{x}_i = 0 \\ & x_i \geq 0, w_i = 0, \quad i \in I \backslash (I_x \cap I_w) \text{ and } \bar{w}_i = 0 \\ & x_i \geq 0, w_i = 0, \quad i \in L \\ & x_i = 0, w_i \geq 0, \quad i \in (I_x \cap I_w) \backslash L \\ & w_j = 0, \quad j \in \{1, \dots, n\} \backslash I. \end{array}$$

This concept is very important since any local and global minimum for the LPLCC is a B-stationary point [17, 53, 55]. However, a certificate for B-stationarity has a combinatorial nature and may be quite demanding for degenerate solutions with $|I_x \cap I_w|$ being relatively large. On the other hand, a strongly stationary point is much more accessible for embedding within an algorithm. A solution $(\bar{x}, \bar{y}, \bar{w})$ of the GLCP is a *strongly stationary point* [17] if and only if there exist $\lambda \in \mathbb{R}^n$, $v \in \mathbb{R}^t$ $\alpha, \beta \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}^p$ such that

$$\begin{aligned} & 0 = E^T \lambda + \alpha \\ & \nabla_x f(\bar{x}, \bar{y}) = -M^T \lambda + \beta \\ & \nabla_y f(\bar{x}, \bar{y}) = -N^T \lambda + A^T v + \gamma \\ & \alpha_i \geq 0,\ \beta_i \geq 0,\ i \in I_x \cap I_w \\ & \alpha_i \bar{w}_i = \beta_i \bar{x}_i = 0,\ i \in I \\ & \beta_j = 0,\ j \in \{1, \dots, n\} \backslash I \\ & \gamma \geq 0 \\ & \gamma^T \bar{y} = 0, \end{aligned}$$

where $\nabla_x f(\bar{x}, \bar{y})$ and $\nabla_y f(\bar{x}, \bar{y})$ represent the vectors of components of the gradient of f at $(\bar{x}, \bar{y}, \bar{w})$ associated with the x- and y-variables. For the LPLCC, $f(\bar{x}, \bar{y}) = c^T \bar{x} + d^T \bar{y}$ and

$$\nabla_x f(\bar{x}, \bar{y}) = c, \qquad \nabla_y f(\bar{x}, \bar{y}) = d.$$

It follows from the definitions that any strongly stationary point is a B-stationary point [17, 53, 55]. The converse is valid for nondegenerate solutions or under a linear independence constraint qualification (LICQ) discussed in [53, 55]. This algebraic definition of a strongly stationary point and the ability to compute a solution of the GLCP discussed in Section 3, enable the development of a complementarity active-set (CASET) algorithm for the MPLCC and LPLCC. Such a procedure was initially proposed in [56] and was subsequently improved, implemented, and tested in [38]. The algorithm starts with a solution of the GLCP and uses an active-set methodology that maintains the complementarity constraint to be satisfied throughout the process. This is achieved by simple modifications of the criteria for the selection of the constraints to be removed and inserted in the working active-set in each iteration. As discussed in [38], the algorithm achieves global convergence to a strongly stationary point under the LICQ condition mentioned before. Recently [17], this algorithm has been extended to guarantee a B-stationary point in the degenerate case.

The CASET algorithm was implemented using MINOS environment [38, 47], and the new ideas discussed in [17] can also be incorporated in this implementation. Computational experiments reported in [38] show that the CASET algorithm is usually quite efficient for finding a strongly stationary point. Computing a B-stationary point in the degenerate case is much more demanding in practice. However, numerical results presented in [17] clearly indicate that the complementarity active-set method performs well in practice even for degenerate solutions, and seems to outerperform other interesting alternative nonlinear programming (penalty, regularization, smoothing, nonsmooth, interior-point and SQP) approaches that have been designed for finding a stationary point of a mathematical program with (linear or nonlinear) complementarity constraints, and can also be applied to the LPLCC [2, 3, 9, 15, 16, 19, 20, 21, 22, 28, 31, 32, 43, 44, 51, 53, 57].

5 A sequential algorithm for finding a global minimum of LPLCC

The algorithm was introduced in [35] for computing a global minimum of a linear bilevel program by exploiting its equivalence to an LPLCC, and was subsequently extended to deal with the general LPLCC [34, 36]. It contains two procedures, namely a parametric method (PAREN) for finding solutions of the GLCP with strictly decreasing objective function values, and the CASET algorithm for finding strongly stationary points of the LPLCC. In order to describe the algorithm, consider the LPLCC (1)-(2) and let K_y be defined by (16). In each major iteration k of the sequential method, the linear objective function is incorporated in a constraint of the form $c^T x + d^T y \leq \lambda_k$, where λ_k is a parameter that prevents the computation of a solution of the GLCP with objective value equal or superior to that for the previously computed solutions. The introduction of this additional constraint leads to another GLCP of the form

$$
\begin{array}{ll}
GLCP(\lambda_k): & \begin{bmatrix} Ew \\ 0 \\ \mu \end{bmatrix} = \begin{bmatrix} q \\ b \\ \lambda_k \end{bmatrix} + \begin{bmatrix} M & N \\ 0 & -A \\ -c^T & -d^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
& y, \mu \geq 0 \\
& x_i \geq 0,\ w_i \geq 0,\ i \in I \\
& w_j = 0, j \in \{1, \ldots, n\} \backslash I \\
& w_i x_i = 0,\ i \in I.
\end{array}
\tag{28}
$$

In the initial iteration ($k = 1$), the algorithm finds a solution $(\bar{x}, \bar{y}, \bar{w})$ to the original GLCP via one of the procedures discussed in Section 3. Starting with this initial point $(\bar{x}, \bar{y}, \bar{w})$, the CASET algorithm is then applied to find a strongly stationary point for the LPLCC. Any subsequent iteration $k > 1$

starts by updating the parameter λ_k according to

$$\lambda_k = c^T x^{k-1} + d^T y^{k-1} - \gamma \, |c^T x^{k-1} + d^T y^{k-1}|, \tag{29}$$

where $(x^{k-1}, y^{k-1}, w^{k-1})$ is the last stationary point found by the CASET algorithm and where γ is a small positive parameter. If the GLCP (λ_k) has a solution, then, as before, the CASET algorithm is employed to obtain a new strongly stationary point for LPLCC. If GLCP has no solution, then $(x^{k-1}, y^{k-1}, w^{k-1})$ is at least an ε-approximate global minimum with $\varepsilon = \gamma \, |c^T x^{k-1} + d^T y^{k-1}|$.

The algorithm is therefore an intelligent procedure for searching for stationary points of the LPLCC until an ε-approximate global minimum is computed. In practice, the algorithm is able to find a global minimum even for not too small values of γ [34, 35, 36].

As discussed in the previous section, the extended procedures described in [17] can be employed in the CASET algorithm in order to guarantee a strongly stationary or a B-stationary point in each major iteration of the algorithm. In our opinion, a certificate of B-stationarity is quite demanding for the objective of deriving during each major iteration a solution of the GLCP that has an objective function value smaller than the one obtained in the previous iteration. However, the investigation on the use of these extended procedures in the sequential complementarity algorithm is an interesting topic for future research.

Next, we discuss the solution of the GLCP(λ_k) for each iteration $k > 1$. As shown in [39], each GLCP(λ_k) is NP-hard and only an enumerative method is able to always solve it in practice. Numerical results reported in [34, 35, 36] clearly indicate that the enumerative method discussed in Section 3 is a valid approach to deal with all the solvable cases of GLCP(λ_k) as required by the sequential method. To understand this, recall that solving the GLCP(λ_k) amounts to finding a global minimum of the following nonconvex quadratic program:

$$\begin{array}{ll} Minimize & \sum_{i \in I} x_i w_i \\ subject\, to & linear \;\; constraints \;\; of \; (28). \end{array} \tag{30}$$

For a solvable GLCP(λ_k), the global minimum value of the objective function is zero and this serves as a stopping criterion for the enumerative method. Since the CASET algorithm is usually fast in finding a strongly stationary point of the LPLCC and the number of major iterations (number of GLCPs(λ_k) to be solved) is usually small in practice, then the sequential algorithm is usually efficient in computing a global minimum for a LPLCC. However, to provide a certificate that such a global minimum has been achieved, it is required to establish that the last GLCP(λ_k) has no solution. This amounts to showing that the quadratic program (30) has a global minimum with a positive objective function value. Hence no premature stopping criterion can be used and the enumerative method typically requires an extensive tree search before terminating with such a certificate.

The design of an efficient procedure to provide a certificate of global optimality has been the subject of intense research. An interesting approach is to design an underestimating optimization problem whose global minimum is relatively easy to compute and that yields a positive lower bound for the program (30). Then GLCP(λ_k) can be declared as unsolvable. In particular SDP [10] and RLT [59] techniques may be useful in this extent. Despite promising results in some cases, much research has to be done to assure the general efficiency of these techniques in practice.

6 Branch-and-bound algorithms for a global minimum of LPLCC

Consider again the LPLCC (1)-(2). Similar to the enumerative method, a branch-and-bound algorithm for the LPLCC exploits a binary tree of the form presented in Section 3, which is constructed based on the dichotomy that $x_i = 0$ or $w_i = 0$ for the pairs of complementary variables. The simplest technique of this type has been introduced by Bard and Moore in [7] for finding a global minimum of a linear bilevel program by exploiting its LPLCC formulation. This method can be applied to the general LPLCC without any modification. For each node of the binary tree generated by the branch-and-bound algorithm, a lower bound for the optimal value of the LPLCC is computed by solving the so-called relaxed linear program RLP(k) that is obtained from the LPLCC (1)-(2) by omitting the complementarity constraints and adding some equalities $z_i = 0$, where z_i is an x_i- or w_i-variable that was fixed along the branches on the path from the root to the current node k. For instance, the RLP(5) associated with node 5 of the binary tree of Section 3 takes the following form:

$$\begin{array}{rl} Minimize & c^T x + d^T y \\ subject\,to & Ew - Mx - Ny = q \\ & y \in K_y \\ & w_i \geq 0,\ x_i \geq 0,\ i \in I \\ & w_j = 0,\ j \in \{1, \ldots, n\} \backslash I \\ & x_{i_1} = 0,\ w_{i_2} = 0. \end{array}$$

If the optimal solution $(\bar{x}, \bar{y}, \bar{w})$ obtained for this RLP(k) satisfies the complementarity constraints, then $c^T \bar{x} + d^T \bar{y}$ is an upper-bound for the global optimal value of the LPLCC. The tree is then pruned at the node k and a new open node is investigated. If $(\bar{x}, \bar{y}, \bar{w})$ is not a complementary solution, then there must exist at least an index i such that $\bar{x}_i > 0$ and $\bar{w}_i > 0$. A branching is then performed from the current node k and two nodes $(k + 1)$ and $(k + 2)$ are generated such that respectively restrict $x_i = 0$ and $w_i = 0$. Termination of the algorithm occurs when there is no open node whose lower bound is smaller than the best upper bound computed by the algorithm. In this case the solution $(\tilde{x}, \tilde{y}, \tilde{w})$ associated with this upper bound is a global minimum for the LPLCC.

The branch-and-bound algorithm should include good heuristics rules for choosing the open node and the pair of complementary variables for branching. The algorithm terminates in a finite number of iterations (nodes) with a global minimum or with a certificate that there is no complementary feasible solution, or that the LPLCC is unbounded. Computational experience reported in [4, 5, 13, 34, 35, 36] indicates that the algorithm is not very efficient for dealing with LPLCC, as the number of nodes tends to greatly increase with the number $|I|$ of pairs of complementary variables.

During the past several years, a number of methodologies have been recommended by many authors to improve the Bard and Moore branch-and-bound algorithm [4, 5, 10, 13, 25, 61]. These improvements have been concerned with the quality of the lower bounds and upper bounds and the branching procedure. Cutting planes [4, 5, 61] and SDP [10] have been used for computing better lower bounds than the ones given by the relaxed linear programs. On the other hand, some ideas of combinatorial optimization have been employed to design more efficient branching strategies that lead to better upper bounds for the branch-and-bound method [4, 5, 13, 25]. Computational experiments reported in [4, 5, 10, 13, 25, 61] clearly indicate that these techniques portend significant improvements for the efficiency of branch-and-bound methods in general.

Another improvement of the Bard and Moore algorithm has been proposed in [37]. The resulting complementarity branch-and-bound (CBB) algorithm can be applied to an LPLCC with a constraint set of the form (18), where H is a PSD matrix, and is therefore useful for finding a global minimum for linear and linear-quadratic bilevel problems and for disjoint bilinear programs. Contrary to the Bard and Moore method, the CBB algorithm uses solutions of the GLCP throughout the process. Therefore, the CASET algorithm can be applied at each node with a significant improvement on the quality of the upper bounds. Disjunctive cuts are recommended to find lower bounds for the LPLCC. Computational experience reported in [37] indicate that the CBB algorithm outerperforms Bard and Moore method in general, and appears to be a promising approach for the computation of a global minimum for the LPLCC.

7 Solution of LPLCC by integer programming

Consider the LPLCC (1) - (2) with K_y given by (16), and let K be the feasible set of the corresponding GLCP. If θ is a positive real number such that

$$\begin{array}{c} Max \\ (x,y,w) \in K \end{array} \quad x_i \leq \theta \ , \qquad \begin{array}{c} Max \\ (x,y,w) \in K \end{array} \quad w_i \leq \theta \ , \tag{31}$$

then each complementarity constraint $x_i w_i = 0$ can be replaced by

$$\begin{array}{l} x_i \leq \theta z_i \\ w_i \leq \theta(1 - z_i) \\ z_i \in \{0, 1\}. \end{array} \tag{32}$$

By applying this transformation to each one of the constraints $x_i w_i = 0$, $i \in I$, the LPLCC reduces to the following mixed-integer linear program (MILP):

$$\begin{array}{ll} Minimize & c^T x + d^T y \\ subject\ to & Ew = q + Mx + Ny \\ & Ay = b \\ & x_i \leq \theta z_i \ , \quad i \in I \\ & w_i \leq \theta(1 - z_i) \ , \quad i \in I \\ & w_j = 0 \ , \quad j \in J \\ & x_i \geq 0 \, , w_i \geq 0 \, , \quad i \in I \\ & z_i \in \{0, 1\} \ , \quad i \in I \\ & y \geq 0, \end{array} \tag{33}$$

where $J = \{ 1, \ldots, n\} \backslash I$. Therefore a global minimum for the LPLCC can be found by computing a global minimum to this MILP. This approach has been used by some authors for finding a global minimum of the LPLCC [24]. It is important to add that such an equivalence also provides a certificate of unsolvability and unboundedness of the LPLCC from those pertaining to the MILP.

An obvious drawback of this approach lies in the existence of the large positive constant θ that may not even exist. A first approach of overcoming such a drawback is to consider θ as a nonnegative variable and solve the resulting mixed-integer nonlinear program (MINLP) by an appropriate technique such as BARON [54, 60]. It is also possible to reduce the LPLCC into a MINLP without the use of such a parameter by exploiting for each $i \in I$ the equivalence between $x_i w_i = 0$ and

$$\begin{array}{l} x_i(1 - z_i) = 0 \\ z_i w_i = 0 \\ z_i \in \{0, 1\}, \end{array}$$

as first established in [45]. As before, the resulting MINLP can be solved by BARON [54, 60] or via any other special-purpose algorithm. This equivalence and the so-called reformulation-linearization technique (RLT) [59] can also be used to construct an MILP formulation of the LPLCC. However, the GLCP has to be solvable and its feasible set must be bounded for the reduction to be possible. Furthermore, the resulting formulation has too many variables that result from the RLT approach. Despite these drawbacks, some promising computational results for LCPs of relatively small dimensions have been reported in [59].

A probably better idea for avoiding the use of a large constant has been introduced in [27] and has been subsequently applied to the special case of LPLCC associated with nonconvex quadratic programs [26]. Consider the MILP as a multiparametric linear program LP(θ, z) on the parameters θ and z. Given any values of θ and z, the dual $DLP(z, \theta)$ of this linear program takes the form:

$$\begin{array}{rl} Maximize & q^T\alpha + b^T\beta - \theta[z^T u + (e-z)^T v_I] \\ subject\ to & (E^T\alpha)_i - v_i \leq 0, \quad i \in I \\ & (E^T\alpha)_j + v_j = 0, \ j \in J \\ & (-M^T\alpha)_i - u_i \leq c_i, \quad i \in I \\ & (-M^T\alpha)_j = c_j, \ j \in J \\ & -N^T\alpha + A^T\beta \leq d \\ & u \geq 0, \ v_I \geq 0, \end{array}$$

where $\alpha \in \mathbb{R}^p$, $\beta \in \mathbb{R}^t$, $u \in \mathbb{R}^{|I|}$ and $v \in \mathbb{R}^n$ are the dual variables associated with the constraints of LP(θ, z), and where $v_I = (v_i)_{i\in I} \in \mathbb{R}^{|I|}$, $z = (z_i)_{i\in I} \in \mathbb{R}^{|I|}$, and $e \in \mathbb{R}^{|I|}$ is a vector of ones. An interesting property of this linear program is that its constraint set does not depend on the values of θ and z. By recognizing this fact and using a minimax integer programming formulation of the MINLP (33), a Benders decomposition technique has been designed in [27] that uses extreme points and unbounded rays of the dual constraint set. This algorithm has been shown to converge in a finite number of iterations with a global minimum of the LPLCC or with a certificate of unsolvability or unboundedness [26, 27]. Simple (or disjunctive) cuts and a recovery procedure for obtaining a solution to the GLCP from a feasible solution are recommended in a preprocessing phase to enhance the efficiency of the algorithm [27]. Computational experiments reported in [26, 27] indicate that the method is in general efficient in practice. Furthermore, the preprocessing phase has a very important impact on the computational performance of the algorithm. The possible use of the sequential complementarity algorithm discussed in Section 5 in the preprocessing phase seems to be an interesting topic for future research.

8 Conclusions

In this paper, we have presented a number of formulations of important nonconvex programs as linear programs with linear complementarity constraints (LPLCC). Algorithms for finding a complementary feasible solution for the LPLCC were discussed. Active-set and interior-point methods, DC and absolute value programming seem to work well for special cases, but not in general. An enumerative method that incorporates a local quadratic solver can efficiently find such a solution in general. A complementarity active set method is recommended for finding a strongly stationary or a B-stationary point for the LPLCC. Computing a global minimum of an LPLCC is a much more difficult task that can be done by either using an enumerative based algorithm that is applied directly to the problem, or by solving an equivalent mixed-integer linear or nonlinear program. Despite the promising numerical performance of these techniques for finding a complementary feasible solution, a stationary point, and a global minimum for the LPLCC, the design of more efficient methodologies and better certificates for a global minimum are important topics for future research.

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