Copositivity and constrained fractional quadratic problems

Paula Amaral * Immanuel M. Bomze † Joaquim Júdice ‡

September 7, 2012

Abstract

We provide Completely Positive and Copositive Optimization formulations for the Constrained Fractional Quadratic Problem (CFQP) and Standard Fractional Quadratic Problem (StFQP). Based on these formulations, Semidefinite Programming (SDP) relaxations are derived for finding good lower bounds to these fractional programs, which can be used in a global optimization branch-and-bound approach. Applications of the CFQP and StFQP, related with the correction of infeasible linear systems and eigenvalue complementarity problems are also discussed.

1 Introduction

Copositive optimization is an emerging field in optimization. The success of this topic is due, not only to the elegance of the theory, but also to the good results obtained in tighter semidefinite relaxations for hard combinatorial optimization problems. For recent papers with a survey flavor see, e.g., [12, 18, 22], and for a clustered bibliography [14].

The lower bounds based on this technique can be favorably compared with bounds obtained by other methods. For instance, a study with the standard quadratic problem reveals the dominance of copositivity based bounds over alternative techniques [11]. Although copositive matrices have been studied for long in linear algebra [27], direct applications in optimization are relatively recent. The idea of reformulating some combinatorial optimization problems, such as the maximum clique problem, as an optimization problem over the copositive cone was first proposed in [9]. This reformulation does not, of course, drain out the difficulty of the problems. The hard component of the optimization problems is cast into a feasibility condition with respect to the copositive or completely positive cone, allowing for a remaining linear representation of the problem. The major drawback has to do

*DM, Universidade Nova de Lisboa, Portugal.
†ISDS, University of Vienna, Austria.
‡DM, Universidade de Coimbra and Instituto de Telecomunicações, Coimbra, Portugal.

with algorithmic aspects. Verification of copositivity or complete positivity is (co-)NP-hard [35, 20]. While more results for dealing with this problem are emerging, such as those of Bundfuss and Dür for adaptive approximations of the copositive cone [16], and copositivity detection [15, 10], the existent theory already enables and justifies copositivity-based approaches. Computational results certify this statement.

In this paper, we consider the problem of minimizing a fractional problem involving the ratio of two quadratic functions, over a polytope. The challenge in addressing this problem arises from the nonconvexity of the objective function, while the motivation lies on its many applications, such as the Constrained Total Least Squares Problem (CTLSP). The unconstrained Total Least Squares Problem (TLSP) is concerned with the Least Squares Problem (LSP) with the additional assumption of corruptness of the data as well as the output. The CTLSP is a TLSP with additional constraints. There are some important subclasses of the CTLSP, such as the Regularized Total Least Squares Problem (RTLSP), where an additional quadratic constraint (Tikhonov regularization) is considered to ensure solution stability. The application of Tikhonov regularization to the TLS problem was introduced by Golub, Hansen and O'Leary [23], where a parameter-dependent direct algorithm for an augmented Lagrangian formulation was proposed. Most of the efficient methods to solve this problem appeared in the last ten years. Simma, Van Huffel and Golub [45] presented an iterative computational approach based on the solution of a quadratic eigenvalue problem (QEP) in each iteration. In [41] an approach also based on an eigenproblem for the RTLSP is solved by an iterative inverse power method. Later the authors improved their work using an alternative derivation of the eigenproblem that allowed the construction of more efficient algorithmic approaches [42]. As pointed out by Beck, Ben-Tal and Teboulle in [7], those methods are guaranteed only to converge to a point satisfying first order necessary optimality conditions. In the paper, the authors presented a parameterized $\varepsilon-$optimal method consisting of the solution of a sequence of convex minimization problems.

There is a generalization of the TLSP that is related with the minimal correction of inconsistent linear systems. In particular, when the minimal correction is defined by the minimization of the Frobenius norm of the perturbations of the matrix of coefficients and the independent term, then this problem can be formulated as a fractional quadratic program (FQP) [2]. When only equalities exist, then the problem is equivalent to the TLSP [24]. The introduction of inequalities in the linear system makes the problem much harder [1]. A branch-and-bound approach was introduced for such a purpose in [2], which includes a Reformulation Linearization Technique (RLT) for finding lower bounds.

Another interesting application of the FQP is the Eigenvalue Complementary Problem (EiCP) with symmetric real matrices. Finding a complementary eigenvalue reduces to finding a stationary point of the Rayleigh quotient on the simplex [40]. Hence, the computation of the largest complementary eigenvalue is equivalent to finding a global minimum of a Standard Fractional Quadratic Program (StFQP). This problem has several applications in engineering and physics, as for instance, in the study of resonance frequency of structures and stability of dynamical systems [19].

Regarding other related contributions, Beck and Teboulle [8] suggested a convex optimization approach for minimizing the ratio of indefinite quadratic functions over an ellipsoid. A main result of the paper is that, under some conditions the problem can be recast as a semidefinite optimization problem with no gap, whose optimal solution can be used to extract the optimal solution of the original problem. This problem can be seen as a generalization of the RTLSP, as the assumptions regarding the quadratic forms in the objective function are mild, but in order to guarantee the existence of a minimum, the matrix of the constraint set must be non-singular. However, in a general RTLSP this matrix is not even necessarily square.

For the general quadratic fractional problem, Gotoh and Konno [25] were able to globally solve small-scale problems using a method that combines the classical Dinkelbach method and a branch-and-bound approach for the nonconvex quadratic problem. Yamamoto and Konno [48] proposed an exact algorithm combining the classical Dinkelbach approach and an integer optimization formulation for solving a nonconvex quadratic optimization problem.

Quadratically constrained quadratic problems are equivalent to a particular subclass of constrained fractional quadratic problems [39]. In fact, if B is a symmetric *positive-definite* (pd) matrix then the problem

$$\min \left\{ \frac{\mathbf{x}^\top C\mathbf{x}}{\mathbf{x}^\top B\mathbf{x}} : A\mathbf{x} = \mathbf{o}\,,\ \mathbf{x} \in \mathbb{R}^n_+ \right\}$$

is equivalent to

$$\min \left\{ \mathbf{y}^\top C\mathbf{y} : A\mathbf{y} = \mathbf{o}\,,\ \mathbf{y}^\top B\mathbf{y} = 1\,,\ \mathbf{y} \in \mathbb{R}^n_+ \right\}\ .$$

In this context it is appropriate to refer some of the recent work on quadratically constrained quadratic problems [39], [6], [33], [3] as valid approaches for the FQP. However, it seems that departure from homogeneity in the constraints $A\mathbf{x} = \mathbf{o}$, that is, considering $A\mathbf{x} = \mathbf{a}$ with $\mathbf{a} \in \mathbb{R}^m \setminus \{\mathbf{o}\}$ instead, yields more complications, at least if $m > 1$. Studying this latter type of problem is the main purpose of the present paper.

To the best of our knowledge, Preisig's article [39] is the only reference where copositivity is explicitly used for finding the global solution to the FQP. This paper deals with the Standard FQP (StFQP, where the feasible set is the standard simplex) and contains two algorithms; the first is a basic line search procedure which uses an unspecified copositivity test as a subroutine, and seems to be not very effective even for medium-scaled problems ($n \geq 20$). This procedure requires basically only strict copositivity of B. To also cope with larger problem dimensions ($n \leq 200$), Preisig suggests in [39] an iterative procedure for which convergence to a KKT point of the StFQP can be proved, provided that B is both *positive-semidefinite* (psd) and strictly copositive. However, no information was provided on the quality of the solution found by this algorithm, and thus even for StFQP this method cannot be considered complete from a global optimization perspective.

1.1 Contributions of the paper

Following the ideas presented in [9] for finding a global minimum of a quadratic nonconvex program over the standard simplex, in this paper an exact completely positive formulation for the CFQP is first introduced. The completely positive condition is relaxed, and a convex semidefinite lower bounding problem is obtained. We prove that dual attainability is impossible for this formulation, and we propose a second dual formulation, based on a more general cone, for which this property is verified. Applications of the CFQP and in particular of the StFQP on the correction of linear systems and symmetric eigenvalue complementarity problem are discussed. Preliminary computational experience with a set of randomly generated CFQPs is reported which illustrates the quality of the lower-bounds as compared with those given by a more traditional approach, such as `BARON` [43]. We also compare our approach with the performance of `GloptiPoly 3`, a general-purpose SDP-based method to optimize rational functions over a semi-algebraic set.

1.2 Outline of the paper

The paper is organized as follows. In Section 2 we introduce the Constrained Fractional Quadratic Problem over a polytope, CFQP, along with some model properties and assumptions.

An exact *Completely Positive (CP) Optimization* formulation for the CFQP, some theoretical results regarding primal and dual attainability and a SDP relaxation based on the CP formulation are discussed in Section 3.

Section 4 studies the Standard Quadratic Fractional Problem (StFQP), that is, a CFQP whose constraint set is the unit simplex. The interest of this study is corroborated by the description of two particular applications of the StFQP, namely the Eigenvalue Complementary Problem (EiCP) and the Constrained Total Least Squares (CTLS). Dimensionality reduction, dual attainability results and lower bounding problems are also discussed in this section.

Computational experience showing the quality of the lower-bounds, of the SDP relaxation of the conic formulation is reported in Section 5. Finally, Section 6 contains some conclusions.

1.3 Notation, matrix cones and duality

Vectors are denoted by lowercase boldface letters (e.g., $\mathbf{o}$ is the zero vector) and matrices by uppercase letters (e.g., O is the zero matrix, or I_n the $n \times n$ identity matrix, the columns of which are denoted by

$\mathbf{e}_1, \dots, \mathbf{e}_n$). $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{R}^n$ denotes n-dimensional Euclidean space and $\mathbb{R}^n_+$ the positive orthant therein, and the standard simplex is denoted by

$$\Delta = \mathrm{conv}\,(\mathbf{e}_1, \dots, \mathbf{e}_n) = \left\{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{e}^\top \mathbf{x} = 1\right\} \tag{1}$$

with $\mathbf{e} = \sum_i \mathbf{e}_i = [1, \dots, 1]^\top \in \mathbb{R}^n$. The notation $A \succeq B$ is used for the condition that $A - B$ is psd, while $A \geq B$ means that $A - B$ has no negative entries. The transpose of A is $A^\top$ and $A \bullet B = \mathrm{trace}(AB)$ represents the Frobenius inner product of two matrices A and B in

$$\mathcal{M}_n = \left\{A \text{ an } n \times n \text{ matrix} : A^\top = A\right\} .$$

With respect to this duality, the dual cone of the copositive matrices

$$\mathcal{C}_n = \left\{C \in \mathcal{M}_n : \mathbf{x}^\top C \mathbf{x} \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n_+\right\}$$

is the cone of completely positive matrices

$$\mathcal{C}_n^* = \left\{D \in \mathcal{M}_n : D = YY^\top, Y \text{ an } n \times k \text{ matrix with } Y \geq O\right\} .$$

Let $\mathcal{P}_n \subset \mathcal{M}_n$ be cone of symmetric psd $n \times n$ matrices and $\mathcal{N}_n \subset \mathcal{M}_n$ be the cone of nonnegative symmetric matrices. It is known that $\mathcal{K}_0 = \mathcal{P}_n + \mathcal{N}_n$ provides a approximation of the copositive cone $\mathcal{C}_n$ in the sense of $\mathcal{K}_0 \subseteq \mathcal{C}_n$. Since $\mathcal{P}_n$ and $\mathcal{N}_n$ are self-dual cones we have

$$\mathcal{C}_n^* \subseteq \mathcal{K}_0^* = (\mathcal{P}_n + \mathcal{N}_n)^* = \mathcal{P}_n \cap \mathcal{N}_n .$$

The latter matrix cone is also called the cone of doubly nonnegative matrices, and sometimes denoted by $\mathcal{D}_n$. Given a general closed, pointed convex cone $\mathcal{K} \subseteq \mathcal{M}$ and its dual cone

$$\mathcal{K}^* = \{S \in \mathcal{M} : S \bullet Z \geq 0 \text{ for all } Z \in \mathcal{K}\}$$

the following programs form a pair of primal-dual conic optimization problems :

$$\max\{C \bullet X : A_i \bullet X = b_i\,,\ 1 \leq i \leq m\,,\ X \in \mathcal{K}^*\} \tag{2}$$

and

$$\min\left\{\mathbf{b}^\top \mathbf{y} : C - \sum_{i=1}^m y_i A_i \in \mathcal{K}\right\} . \tag{3}$$

We will mostly deal with the cases $\mathcal{K} = \mathcal{C}_n$ and $\mathcal{K} = \mathcal{D}_n^* = \mathcal{P}_n + \mathcal{N}_n$, but any choice $\mathcal{K} = \mathcal{K}_n^r$ for usual SDP- or LP-based approximation hierarchies $(\mathcal{K}_n^r)_{r \in \mathbb{N}}$ would do, where $\mathcal{K}_n^r$ is in some sense close to $\mathcal{C}_n$ for large r; see [37, 13, 38, 26, 49, 21], who all more or less follow the ideas first put forward in [36, 32]. Recall that checking membership of $\mathcal{K}_n^r$ in any such hierarchy usually involves psd matrices of order n^{r+1}, rendering these approximations computationally intractable for large r and n.

2 The Constrained Fractional Quadratic Problem

2.1 Problem formulation and model assumptions

In this section we consider the CFQP

$$\psi = \min \left\{ f(\mathbf{x}) = \frac{\mathbf{x}^\top C\mathbf{x} + 2\mathbf{c}^\top \mathbf{x} + \gamma}{p(\mathbf{x})} : \mathbf{x} \in \mathcal{T} \right\} \tag{4}$$

where

$$\mathcal{T} = \left\{ \mathbf{x} \in \mathbb{R}^n_+ : A\mathbf{x} = \mathbf{a} \right\} \quad \text{and} \quad p(\mathbf{x}) = \mathbf{x}^\top B\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + \beta\,.$$

For simplicity of exposition, let us assume here that there are $0 < \delta < \eta < +\infty$ such that

$$p(\mathbf{x}) \in [\delta, \eta] \quad \text{for all } \ \mathbf{x} \in \mathcal{T}\,. \tag{5}$$

For an in-depth discussion of this and related conditions occurring, e.g., in [29], we refer to the next subsection.

The *Standard Quadratic Optimization Problem* (StQP),

$$\min \left\{ f(\mathbf{x}) = \mathbf{x}^\top C\mathbf{x} : \mathbf{x} \in \Delta \right\} \tag{6}$$

is a special case of the CFQP when $p(\mathbf{x}) \equiv 1$ and the polytope $\mathcal{T} = \Delta$ is the standard simplex. This problem is known to be NP-hard and thus the same applies to the CFQP (4).

For convenient notation, we introduce

$$\overline{A} = \begin{bmatrix} \mathbf{a}^\top \mathbf{a} & -\mathbf{a}^\top A \\ -A^\top \mathbf{a} & A^\top A \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} \beta & \mathbf{b}^\top \\ \mathbf{b} & B \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} \gamma & \mathbf{c}^\top \\ \mathbf{c} & C \end{bmatrix}. \tag{7}$$

Using Schur complements, it is easy to see that condition (5) holds if $\beta B - \mathbf{b}\mathbf{b}^\top$ is pd and $\mathcal{T}$ is bounded, but (5) may hold in relevant cases even if $\beta B - \mathbf{b}\mathbf{b}^\top$ is singular; see Subsection 4.3 below.

Note that $\overline{A} = [-\mathbf{a}\,,\, A]^\top [-\mathbf{a}\,,\, A] \in \mathcal{P}_{n+1}$ is psd but, typically, singular:

$$A\mathbf{x} = \mathbf{a} \quad \iff \quad \overline{A}\mathbf{z} = [-\mathbf{a}\,,\, A]\mathbf{z} = \mathbf{o} \quad \iff \quad \mathbf{z}^\top \overline{A}\mathbf{z} = 0\,,$$

where $\mathbf{z} = [1\,,\, \mathbf{x}^\top]^\top \in \mathbb{R}^{n+1}$. Hence we may rephrase (4) as

$$\psi = \min \left\{ \frac{\mathbf{z}^\top \overline{C}\mathbf{z}}{\mathbf{z}^\top \overline{B}\mathbf{z}} : \mathbf{z} \in \mathbb{R}^{n+1}_+\,,\ z_1 = 1\,,\ \mathbf{z}^\top \overline{A}\mathbf{z} = 0 \right\}. \tag{8}$$

Problems of this kind appear in context of repair of inconsistent linear (inequality) systems, see Subsection 4.3 below. In the sequel, we will always assume $\overline{A} \neq O$, which implies $\operatorname{trace}(\overline{A}) > 0$.

The feasible set $\mathcal{T}$ is compact if and only if $\mathcal{T} \neq \emptyset$ and

$$\ker A \cap \mathbb{R}^n_+ = \{\mathbf{o}\}\,,$$

which amounts to require that $A\mathbf{y} = \mathbf{o}$ and $\mathbf{y} \in \mathbb{R}^n_+$ together already imply $\mathbf{y} = \mathbf{o}$. Further, we introduce the polyhedral cone generated by the constraints

$$\Gamma_{\overline{A}} = \left\{ \mathbf{z} \in \mathbb{R}^{n+1}_+ : \overline{A}\mathbf{z} = \mathbf{o} \right\}. \tag{9}$$

As usual, we say that $\overline{B}$ is *strictly* $\Gamma_{\overline{A}}$*-copositive* if and only if $\mathbf{z} \in \mathbb{R}^{n+1}_+ \setminus \{\mathbf{o}\}$ and $\overline{A}\mathbf{z} = \mathbf{o}$ imply $\mathbf{z}^\top \overline{B}\mathbf{z} > 0$.

Lemma 1 *If $\mathcal{T}$ is compact, strict positivity of p over $\mathcal{T}$ is equivalent to strict $\Gamma_{\overline{A}}$ -copositivity of $\overline{B}$, and this implies condition (5).*

Proof. If $\mathbf{z} = \begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix}$ then $\mathbf{z}^\top \overline{B}\mathbf{z} = p(\mathbf{x})$ and $\mathbf{z} \in \Gamma_{\overline{A}}$ implies that $\mathbf{x} \in \mathcal{T}$. Hence strict $\Gamma_{\overline{A}}$ -copositivity of $\overline{B}$ is sufficient for positivity of p over $\mathcal{T}$. To see necessity, let $\mathbf{z} = \begin{bmatrix} \zeta \\ \mathbf{v} \end{bmatrix} \in \Gamma_{\overline{A}}$ with $\zeta > 0$. Then $\mathbf{x} = \frac{1}{\zeta}\mathbf{v} \in \mathcal{T}$ and $\mathbf{z}^\top \overline{B}\mathbf{z} = \zeta^2 p(\mathbf{x}) > 0$. However, if $\zeta = 0$, then $\mathbf{v} \in \mathbb{R}^n_+$ must satisfy $A\mathbf{v} = \mathbf{o}$, by the construction of $\overline{A}$. Hence $\mathbf{v} = \mathbf{o}$ and strict $\Gamma_{\overline{A}}$ -copositivity of $\overline{B}$ follows. $\square$

Compactness of $\mathcal{T}$ and strict positivity of p over this set implies that problem (4) always has an optimal solution (primal attainability).

For further convenient reference, we repeat our overall model assumptions here:

$$\left.\begin{array}{l} \mathcal{T} = \left\{\mathbf{x} \in \mathbb{R}^n_+ : A\mathbf{x} = \mathbf{a}\right\} \neq \emptyset\,; \\ \ker\, A \cap \mathbb{R}^n_+ = \{\mathbf{o}\} \quad \iff \quad A\mathbf{y} \neq \mathbf{o} \text{ if } \mathbf{y} \in \mathbb{R}^n_+ \setminus \{\mathbf{o}\}\,; \\ \overline{B} \text{ is strictly } \Gamma_{\overline{A}}\text{-copositive: } \; \mathbf{z}^\top \overline{B}\mathbf{z} > 0 \text{ if } \overline{A}\mathbf{z} = \mathbf{o}\,,\, \mathbf{z} \in \mathbb{R}^n_+ \setminus \{\mathbf{o}\}\,. \end{array}\right\} \tag{10}$$

2.2 SDP approach for general rational optimization

In the paper [29], SDP-based methods for optimization of general rational polynomial functions $f(\mathbf{x}) = \frac{n(\mathbf{x})}{p(\mathbf{x})}$ with polynomials $n(\mathbf{x})$ and $p(\mathbf{x})$ over feasible sets S are studied, where either $S = \mathbb{R}^n$ (the unconstrained case) or else S is a semi-algebraic set which is the (partial) closure of an open set.

Here we are dealing with the constrained case $\min\{f(\mathbf{x}) : \mathbf{x} \in \mathcal{T}\}$, where n and p are of degree two, but typically $\mathcal{T}$ has no interior points. So the assumptions on $\mathcal{T}$ here and on S in [29] are incompatible. But there are further assumptions on the problems which need discussion. Before proceeding to them, note that a closer look at the arguments in [29] reveals that the above-mentioned assumption on S can be replaced with the following assumption on S and p:

$$\left.\begin{array}{l} \text{for any pair } \{\mathbf{x}_+, \mathbf{x}_-\} \subseteq S\,, \text{ there is a path } \mathbf{x}(t) \in S \\ \text{linking } \mathbf{x}_+ \text{ and } \mathbf{x}_-\,, \text{ i.e., } \mathbf{x}(0) = \mathbf{x}_- \text{ and } \mathbf{x}(1) = \mathbf{x}_+\,, \\ \text{such that } \chi(t) = p(\mathbf{x}(t)) \text{ is a polynomial in } t\,. \end{array}\right\} \tag{11}$$

Obviously, this condition is satisfied if S is a convex set, a property our feasible set $\mathcal{T}$ enjoys, and is also implied if existence of polynomial paths $\mathbf{x}(t)$ linking any two points in S is guaranteed. A study of this latter condition falls into the field of real algebraic geometry and therefore is beyond the scope of this paper. So let us proceed to two further assumptions stated in [29]:

(a) the polynomials p and n have no common real polynomial factor which is non-constant;

(b) the polynomials p and n have no common (real) root in S.

It is easily seen that (a) and (b) are not implied by each other, take, e.g., $p(\mathbf{x}) = 2x_1^2 + x_2^2 - 1$ and $n(\mathbf{x}) = x_1^2 + 2x_2^2 - 1$ with S the unit disc (or $\mathcal{T}$ some polytope containing S), or else $n(\mathbf{x}) = 2p(\mathbf{x}) = x_1^2 + x_2^2$ and $S = \mathcal{T} = \Delta$.

Assumption (a) can also be easily enforced for the CFQP. Otherwise we would arrive at the fractional linear case for which of course there is the LP reformulation, going back to Charnes and Cooper (a referee kindly pointed out that there is also a (nonlinear) SDP formulation in [47, Section 7.1]). However, while seemingly quite natural, assumption (a) is not needed in the following auxiliary result which deals with boundedness of the constrained rational optimization problem. Also, S can be an arbitrary set satisfying (11), e.g., any convex set.

Proposition 2.1 *Suppose that (11) holds and that the polynomials p and n have no common (real) root in S. Then $f(\mathbf{x}) = \frac{n(\mathbf{x})}{p(\mathbf{x})}$ can be bounded (from below and/or above) over S only if $p(\mathbf{x})$ does not change sign strictly over S. To be more precise: if $S_0 = \{\mathbf{x} \in S : p(\mathbf{x}) = 0\}$, then*

$$\sup\{f(\mathbf{x}) : \mathbf{x} \in S \setminus S_0\} = +\infty \quad \textit{and} \quad \inf\{f(\mathbf{x}) : \mathbf{x} \in S \setminus S_0\} = -\infty\,,$$

provided that there are $\{\mathbf{x}_+, \mathbf{x}_-\} \subseteq S$ such that $p(\mathbf{x}_-) < 0 < p(\mathbf{x}_+)$.

Proof. Suppose $p(\mathbf{x}_-) < 0 < p(\mathbf{x}_+)$ for some $\{\mathbf{x}_+, \mathbf{x}_-\} \subseteq S$ and link these points by a polynomial path $\mathbf{x}(t) \in S$ as in assumption (11). Then, as $\chi(t) = p(\mathbf{x}(t))$ is a univariate polynomial with $\chi(0) < 0 < \chi(1)$, it is well known (and quite elementary to prove) that there must be a transversal real root $\bar{t} \in [0, 1]$ of χ: $\chi(\bar{t} - \varepsilon) < 0 < \chi(\bar{t} + \varepsilon)$ must hold for sufficiently small $\varepsilon > 0$. Since $\bar{\mathbf{x}} = \mathbf{x}(\bar{t})$ is a root of p in S, we must have $n(\bar{\mathbf{x}}) \neq 0$, by assumption (b) above. Suppose for the moment that $n(\bar{\mathbf{x}}) > 0$; then along the two sequences $\mathbf{x}_\varepsilon^\pm = \mathbf{x}(\bar{t} \pm \varepsilon)$, we have evidently $\lim\limits_{\varepsilon \searrow 0} f(\mathbf{x}_\varepsilon^\pm) = \pm\infty$, and the result follows. The same argument holds in the opposite case $n(\bar{\mathbf{x}}) < 0$, switching signs. $\square$

So apparently the common root assumption (b) plays a key role in investigating boundedness. However, for the CFQP the procedure suggested in [29] to check this, namely to certify

$$\inf\left\{p^2(\mathbf{x}) + n^2(\mathbf{x}) : \mathbf{x} \in S\right\} > 0\,,$$

requires bounding a quartic optimization problem over $\mathcal{T}$ which may be even more difficult than establishing copositivity to enforce the model assumptions (10). For sure there are non-convex instances for this quartic objective function. For instance consider *arbitrary* B and $\mathbf{b}$ with $\beta = 1$ and select any $\mathbf{x}$ in the interior of $\mathbb{R}^n_+$. Next pick a vector $\mathbf{v} \neq \mathbf{o}$ with $\mathbf{v} \perp B\mathbf{x} + \mathbf{b}$, then select A such that $A\mathbf{v} = \mathbf{o}$ but arbitrary else, and put $\mathbf{a} = A\mathbf{x}$ so that $\mathbf{x} \pm t\mathbf{v} \in \mathcal{T}$ for sufficiently small $t > 0$. Now choose $C = B - \rho I$ and $\mathbf{c} = \mathbf{b} + \rho\mathbf{x}$ where $\rho \in \mathbb{R}$ is to be determined later, and $\gamma = 1 - \rho\mathbf{x}^\top\mathbf{x}$, so that $p(\mathbf{x}) = n(\mathbf{x})$, and, by construction, $\mathbf{v} \perp \{B\mathbf{x} + \mathbf{b}, C\mathbf{x} + \mathbf{c}\}$. Finally let ρ be such that $\mathbf{v}^\top[p(\mathbf{x})B + n(\mathbf{x})C]\mathbf{v} = \mathbf{v}^\top(2B - \rho I)\mathbf{v} < 0$. Then a straightforward calculation shows

$$\mathbf{v}^\top D^2[p^2(\mathbf{x}) + n^2(\mathbf{x})]\mathbf{v} = 2\mathbf{v}^\top[p(\mathbf{x})B + n(\mathbf{x})C]\mathbf{v} < 0\,.$$

This remains true for sufficiently small departures from p which yield a non-constant objective f.

3 Copositivity and CFQP

3.1 Completely positive formulation

As stated before, the fractional quadratic problem (4) can be rewritten in homogeneous form (8). Putting $Z = \mathbf{z}\mathbf{z}^\top$, rewriting $\mathbf{z}^\top\overline{A}\mathbf{z} = \overline{A} \bullet Z$, with $\overline{A}$ psd and observing that $Z_{11} = z_1^2$ and $\mathbf{z} \in \mathbb{R}^{n+1}_+$, we have

$$\psi = \min\left\{\frac{\overline{C} \bullet Z}{\overline{B} \bullet Z} : Z_{11} = 1\,,\ \overline{A} \bullet Z = 0\,,\ \operatorname{rank}(Z) = 1\,,\ Z \in \mathcal{C}^*_{n+1}\right\}.\quad(12)$$

By homogeneity, for any Z feasible to (12) we can replace the constraint $Z_{11} = 1$ by $Z_{11} > 0$. We may also define $X = \frac{1}{\overline{B}\bullet Z}\,Z \in \mathcal{C}^*_{n+1}$ which also has rank one with $X_{11} > 0$ and satisfies $\overline{B} \bullet X = 1$, to obtain the following equivalent problem

$$\begin{aligned}\psi = \min \quad & \left\{\overline{C} \bullet X : \overline{B} \bullet X = 1,\, \overline{A} \bullet X = 0,\, \operatorname{rank}(X) = 1,\right.\\ & \left. X_{11} > 0,\, X \in \mathcal{C}^*_{n+1}\right\}\,.\end{aligned}\tag{13}$$

This problem is non-standard in two aspects. First, it includes a strict linear inequality for defining feasibility; second, and probably more familiar in the context of SDP relaxations, it contains a (non-convex) rank-one constraint. Next we prove that we still obtain an equivalent problem by dropping the rank condition and the constraint $X_{11} > 0$, so that (13) turns out to be equivalent to the following problem

$$\min\left\{\overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, X \in \mathcal{C}^*_{n+1}\right\}\,.\tag{14}$$

To prove this statement we must introduce the following lemma, which parallels an important result on the CP representation of mixed-binary quadratic optimization problems [17]; see also [5].

Lemma 2 *Under the model assumptions (10),*

$$\begin{aligned}&\left\{X \in \mathcal{C}^*_{n+1} : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0\right\} =\\ =\ & \operatorname{conv}\left\{\mathbf{z}\mathbf{z}^\top : \mathbf{z} \in \mathbb{R}^{n+1}_+ : z_1 > 0\,,\ \mathbf{z}^\top\overline{B}\mathbf{z} = 1\,,\ \overline{A}\mathbf{z} = \mathbf{o}\right\}.\end{aligned}$$

Proof. The inclusion $\supseteq$ is immediate given the definition of $\mathcal{C}^*_{n+1}$. For the $\subseteq$ part, first note that any $X \in \mathcal{C}^*_{n+1}$ with $\overline{B} \bullet X = 1$ satisfies

$X \neq O$. Let $X \in \mathcal{C}^*_{n+1} \setminus \{O\}$. Then there is the representation

$$X = \sum_{i=1}^{r} \mathbf{y}_i \mathbf{y}_i^\top \quad \text{with} \quad \mathbf{y}_i \in \mathbb{R}^{n+1}_+ \setminus \{\mathbf{o}\}\,, \text{ for all } i\,,$$

for some $r \geq 1$. Since $\overline{A}$ is psd,

$$0 \leq \mathbf{y}_i^\top \overline{A} \mathbf{y}_i \leq \sum_{j=1}^{r} \mathbf{y}_j^\top \overline{A} \mathbf{y}_j = \overline{A} \bullet X = 0 \quad \Longrightarrow \quad \overline{A}\mathbf{y}_i = \mathbf{o}\,. \tag{15}$$

Hence $\mathbf{y}_i \in \Gamma_{\overline{A}} \setminus \{\mathbf{o}\}$ and we can define

$$\lambda_i = \mathbf{y}_i^\top \overline{B} \mathbf{y}_i$$

which is strictly positive by (10) for all $i = 1, \ldots, r$. Let

$$\mathbf{z}_i = \frac{1}{\sqrt{\mathbf{y}_i^\top \overline{B} \mathbf{y}_i}} \mathbf{y}_i \in \Gamma_{\overline{A}}\,.$$

Then by construction $\mathbf{z}_i^\top \overline{B} \mathbf{z}_i = 1$ and $\overline{A}\mathbf{z}_i = \mathbf{o}$. For all i, the first coordinate ζ_i of $\mathbf{z}_i = \begin{bmatrix} \zeta_i \\ \mathbf{v}_i \end{bmatrix}$ must not vanish. Otherwise $\overline{A}\mathbf{z}_i = \mathbf{o}$ would imply $A\mathbf{v}_i = \mathbf{o}$, and $\mathbf{v}_i \in \mathbb{R}^n_+$ would by (10) yield $\mathbf{v}_i = \mathbf{o}$ or $\mathbf{z}_i = \mathbf{o}$ or $\mathbf{y}_i = \mathbf{o}$, which is absurd. Then

$$\sum_{i=1}^{r} \lambda_i = \sum_{i=1}^{r} \mathbf{y}_i^\top \overline{B} \mathbf{y}_i = \sum_{i=1}^{r} \overline{B} \bullet \mathbf{y}_i \mathbf{y}_i^\top = \overline{B} \bullet X = 1\,. \tag{16}$$

So X can be written as $X = \sum_{i=1}^{r} \lambda_i \mathbf{z}_i \mathbf{z}_i^\top$, and the result follows. □

Theorem 1 *Under the model assumptions (10), problems (13) and (14) are equivalent. Moreover, there is always an optimal solution of the form $Z^* = Z^*_{11}\mathbf{z}\mathbf{z}^\top$ to (14) with $\mathbf{z}^\top = [1\,,\, (\mathbf{x}^*)^\top]$ which encodes in $\mathbf{x}^* \in \mathcal{T}$ an optimal solution to (4).*

Proof. Any optimal solution X^* to (14) is a convex combination of rank-one matrices like Z^*, due to Lemma 2. Hence (13) and (14) must have the same minimal objective value by convexity (in fact, linearity) of the objective function. Therefore this and the remaining assertion follow from standard convex optimization arguments. □

3.2 Duality and copositive optimization

By weak duality of (14)

$$\psi \geq \lambda^* = \sup\left\{\lambda : \overline{C} - \lambda \overline{B} - \mu \overline{A} \in \mathcal{C}_{n+1}\right\}. \tag{17}$$

Slater's condition is always violated for (14). Indeed, if $Z \in \operatorname{int} \mathcal{C}^*_{n+1}$ is feasible to (14), then $Z - \alpha I_{n+1} \in \mathcal{C}^*_{n+1}$ for a small $\alpha > 0$, and in particular this matrix is psd. But

$$\overline{A} \bullet (Z - \alpha I_{n+1}) = 0 - \alpha \operatorname{trace}(\overline{A}) < 0\,,$$

is a contradiction to the fact of $\overline{A} \in \mathcal{P}_{n+1} \setminus \{O\}$. Therefore it is not possible to infer strong duality (in particular, dual attainability) from standard arguments. However, under our assumptions, the dual problem is strictly feasible, which implies attainability of the primal (14) (this was already established in Section 2 before) and zero duality gap, that is $\psi = \lambda^*$. To establish this result, we need to introduce another lemma.

Lemma 3 *If a symmetric matrix D is strictly $\Gamma_{\overline{A}}$ -copositive and $\overline{A}$ is psd, then there is $\rho > 0$ such that $D + \rho\overline{A}$ is strictly copositive (w.r.t. the whole $\mathbb{R}^n_+$).*

Proof. Let $\omega = -\min_{\mathbf{x}\in\Delta} \mathbf{x}^\top D\mathbf{x}$. By continuity of the quadratic form and compactness of Δ, there exist $\delta > 0$ and $\eta > 0$ such that $\mathbf{x}^\top D\mathbf{x} \geq \delta$ whenever $\operatorname{dist}(\mathbf{x}, \ker \overline{A}) < \eta$ and $\mathbf{x} \in \Delta$. On the other hand, the set $P = \{\mathbf{x} \in \Delta : \operatorname{dist}(\mathbf{x}, \ker \overline{A}) \geq \eta\}$ is either empty (the trivial case) or compact. In the latter case, by construction, we have $\nu = \min_{\mathbf{x}\in P} \mathbf{x}^\top \overline{A}\mathbf{x} > 0$. Finally let

$$\rho = \max\{1, \frac{2\omega}{\nu}\} \geq 1 > 0\,.$$

Then $\mathbf{x}^\top(D + \rho\overline{A})\mathbf{x} \geq \mathbf{x}^\top D\mathbf{x} \geq \delta > 0$ for all $\mathbf{x} \in \Delta \setminus P$ by the above construction. But for any $\mathbf{x} \in P$ we have $\mathbf{x}^\top(D + \rho\overline{A})\mathbf{x} \geq -\omega + 2\omega = \omega$. Now the result follows if $\omega > 0$. If $\omega \leq 0$ then $\mathbf{x}^\top D\mathbf{x} \geq 0$ and $\mathbf{x}^\top(D + \rho\overline{A})\mathbf{x} \geq 0 + 1\nu = \nu > 0$. Hence $D + \rho\overline{A}$ is strictly copositive. $\Box$

Theorem 2 *Under the model assumptions (10), the dual problem (17) is strictly feasible (i.e., Slater's condition is satisfied). Hence the duality gap is zero, and the primal problem (14) has always an optimal solution, that is, $\psi = \lambda^* = C \bullet Z^*$ for some Z^* feasible to (14).*

Proof. Lemma 1 and Lemma 3 imply that there is a $\rho > 0$ such that $\overline{B} + \rho\overline{A}$ is strictly copositive. By continuity, this is still true for $\gamma\overline{C} + \overline{B} + \rho\overline{A}$ for small $\gamma > 0$. Also, by positive homogeneity, we may divide by γ and still $\overline{C} - \lambda\overline{B} - \mu\overline{A} \in \operatorname{int}\mathcal{C}^*_{n+1}$, where $\lambda = -\frac{1}{\gamma}$ and $\mu = -\frac{\rho}{\gamma}$. $\Box$

For a slightly modified dual program we present an attainability result.

Theorem 3

$$\psi \quad = \quad \max\left\{\lambda : \overline{C} - \lambda\overline{B} \textit{ is } \Gamma_{\overline{A}} \textit{-copositive} \right\}. \tag{18}$$

Proof. Suppose that $\overline{C} - \lambda\overline{B}$ is $\Gamma_{\overline{A}}$-copositive. For any $\mathbf{x} \in \mathcal{T}$, we have $\mathbf{z} = [1\,,\,\mathbf{x}^\top]^\top \in \Gamma_{\overline{A}}$, so that

$$\mathbf{z}^\top\overline{C}\mathbf{z} - \lambda\mathbf{z}^\top\overline{B}\mathbf{z} \geq 0\,.$$

As $\mathbf{z}^\top \overline{B}\mathbf{z} > 0$, this implies $\lambda \leq \frac{\mathbf{z}^\top \overline{C}\mathbf{z}}{\mathbf{z}^\top \overline{B}\mathbf{z}} = f(\mathbf{x})$, and therefore $\lambda \leq \psi$. To establish the result, we consider a solution $\bar{\mathbf{x}}$ to (4) and show that $\bar{\lambda} = \psi = f(\bar{\mathbf{x}})$ satisfies that $\overline{C} - \bar{\lambda}\overline{B}$ is $\Gamma_{\overline{A}}$-copositive. Let $\mathbf{z} = [1\,,\, \mathbf{x}^\top]^\top \in \Gamma_{\overline{A}}$. For any $\mathbf{x} \in \mathcal{T}$

$$\mathbf{z}^\top \overline{C}\mathbf{z} - \bar{\lambda}\mathbf{z}^\top \overline{B}\mathbf{z} = \mathbf{z}^\top \overline{B}\mathbf{z}\,[f(\mathbf{x}) - f(\bar{\mathbf{x}})] \geq 0\,.$$

Now $\Gamma_{\overline{A}}$-copositivity follows as in the proof of Lemma 1. □

To use (18) directly we should have an algorithm for checking $\Gamma_{\overline{A}}$ -copositivity. While there were some algorithms designed for this task, procedures to check classical $\mathbb{R}^n_+$-copositivity are much more popular [16, 10].

The equality (18) would imply dual attainability if we could prove that for a $\Gamma_{\overline{A}}$-copositive matrix D and a psd matrix $\overline{A}$ there is $\rho \in \mathbb{R}$ such that $D + \rho\overline{A}$ is copositive. Unfortunately, this property does not hold, as the following example shows.

Example: Let $D = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ and $\overline{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
Then D is $\Gamma_{\overline{A}}$ -copositive, but there is no $\rho \in \mathbb{R}$ such that $D + \rho\overline{A}$ is a copositive matrix.

Since the primal problem is never strictly feasible, dual attainability is still not established and remains an open question. However, for some special cases it is possible to have dual attainability, as it is the case of the StFQP analyzed in Section 4. For this class, also an approximation result will be shown: it admits a polynomial-time approximation scheme (see Section **??**).

3.3 Lower bounds based on copositive relaxations

Previously we proved that

$$\begin{aligned} \psi &= \min\left\{ f(\mathbf{x}) = \frac{\mathbf{x}^\top C\mathbf{x} + 2c^\top \mathbf{x} + \gamma}{\mathbf{x}^\top B\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + \beta} : A\mathbf{x} = \mathbf{a}, \mathbf{x} \in \mathbb{R}^n_+ \right\} && (19) \\ &= \min\left\{ \overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, X \in \mathcal{C}^*_{n+1} \right\}. && (20) \end{aligned}$$

Checking condition $X \in \mathcal{C}^*_{n+1}$ is (co-)NP-hard [35, 20], but it is possible to exploit this equality to get a lower bound for the CFQP, using the inclusion $\mathcal{C}^*_{n+1} \subseteq \mathcal{D}_{n+1} = \mathcal{P}_{n+1} \cap \mathcal{N}_{n+1}$. So by solving

$$\psi_{\text{cop}} = \min\left\{ \overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, X \in \mathcal{D}_{n+1} \right\}, \qquad (21)$$

we obtain a lower bound for (19). In addition, given that

$$X = \begin{bmatrix} 1 & \mathbf{x}^\top \\ \mathbf{x} & \mathbf{x}\mathbf{x}^\top \end{bmatrix}$$

for any rank-one completely positive matrix X, we may reinforce the lower bound by requiring in addition an upper bound $\mathbf{u}\mathbf{u}^\top$ in the variable X, where the components of $\mathbf{u} \in \mathbb{R}^{n+1}$ are given by

$$u_{i+1} = \max\left\{ x_i : A\mathbf{x} = \mathbf{a}\,,\ \mathbf{x} \in \mathbb{R}^n_+ \right\} \quad \text{for } i = 1, \ldots, n \text{ and } u_1 = 1\,.$$

Hence the following SDP gives a tighter lower bound for the CFQP:

$$\psi^+_{\rm cop} = \min\left\{\overline{C} \bullet X : \overline{B} \bullet X = 1, \overline{A} \bullet X = 0, X \succeq 0,\ 0 \le X \le \mathbf{u}\mathbf{u}^\top\right\} . \quad (22)$$

The remainder of this subsection investigates the boundedness of the feasible set of the relaxed problem (21).

Lemma 4 *Suppose that assumption (10) holds, and*

$$X = \begin{bmatrix} X_{11} & \mathbf{x}^\top \\ \mathbf{x} & Y \end{bmatrix} \in \mathcal{D}_{n+1} .$$

(a) If $X \neq O$ satisfies $\overline{A} \bullet X = 0$ then $X_{11} > 0$.
(b) If $X_{11} > 0$ and $\overline{A} \bullet X = 0$, then $A^\top A \bullet (X_{11}Y - \mathbf{x}\mathbf{x}^\top) = 0$ and $\bar{\mathbf{x}} = \frac{1}{X_{11}}\,\mathbf{x} \in \mathcal{T}$.

Proof. (a) If $X_{11} = 0$, then also $\mathbf{x} = \mathbf{o}$, and $0 = \overline{A} \bullet X = (A^\top A) \bullet Y$. But $Y \in \mathcal{P}_n$ as $X \in \mathcal{D}_{n+1}$. Hence $Y(\mathbb{R}^n) \subseteq \ker A$ and in particular $Y\mathbf{w} \in \ker A$ for all $\mathbf{w} \in \mathbb{R}^n_+$. Since $Y \in \mathcal{N}_n$, then $Y\mathbf{w} \in \mathbb{R}^n_+$, too. Hence $Y\mathbf{w} \in \ker A \cap \mathbb{R}^n_+ = \{\mathbf{o}\}$, or $Y(\mathbb{R}^n_+) = \{\mathbf{o}\}$, which entails $Y = 0$ and thus $X = O$, contradicting the assumption.
(b) Since $X \in \mathcal{P}_{n+1}$, also the Schur complement $X_{11}Y - \mathbf{x}\mathbf{x}^\top \in \mathcal{P}_n$. Therefore we get

$$0 \le (A^\top A) \bullet (X_{11}Y - \mathbf{x}\mathbf{x}^\top) = X_{11}(A^\top A) \bullet Y - \|A\mathbf{x}\|^2$$

and by consequence

$$\begin{aligned} 0 &\le \|X_{11}\mathbf{a} - A\mathbf{x}\|^2 \\ &= \|A\mathbf{x}\|^2 - 2(X_{11}\mathbf{a})^\top(A\mathbf{x}) + \|X_{11}\mathbf{a}\|^2 \\ &\le X_{11}(A^\top A) \bullet Y - 2(X_{11}\mathbf{a})^\top(A\mathbf{x}) + \|X_{11}\mathbf{a}\|^2 \\ &= X_{11}(\overline{A} \bullet X) = 0\,, \end{aligned}$$

which establishes both assertions. □

For the next auxiliary result we resort on the condition $B \in \mathcal{P}_n$ which was also employed in [39].

Lemma 5 *Assume (10) B psd. Then there is a finite $M > 0$ such that $X_{11} + \|\mathbf{x}\| \le M$ for all X feasible to (21).*

Proof. Suppose that $X^\nu_{11} \nearrow \infty$ along a sequence X_ν of (21)-feasible points. Since $\mathcal{T}$ is compact, we may assume without loss of generality that $\bar{\mathbf{x}}_\nu = \frac{1}{X^\nu_{11}}\mathbf{x}_\nu \to \bar{\mathbf{x}} \in \mathcal{T}$ as $\nu \to \infty$. Since we have

$$1 = X_{11} + 2\mathbf{b}^\top\mathbf{x} + B \bullet Y \ge X_{11} + 2\mathbf{b}^\top\mathbf{x} + \frac{1}{X_{11}}\mathbf{x}^\top B\mathbf{x}$$

due to the fact that both B and the Schur complement $Y - \frac{1}{X_{11}}\,\mathbf{x}\mathbf{x}^\top$ are psd for any feasible X, it follows for $\bar{\mathbf{z}}_\nu = [1, \bar{\mathbf{x}}_\nu^\top]^\top \in \Gamma_{\overline{A}} \setminus \{\mathbf{o}\}$ that

$$\frac{1}{X^\nu_{11}} \ge 1 + 2\mathbf{b}^\top\bar{\mathbf{x}}_\nu + (\bar{\mathbf{x}}_\nu)^\top B(\bar{\mathbf{x}}_\nu) = (\bar{\mathbf{z}}_\nu)^\top\overline{B}(\bar{\mathbf{z}}_\nu)$$

for all ν. Hence in the limit $\bar{\mathbf{z}}^\top \overline{B} \bar{\mathbf{z}} = 0$, contradicting $\bar{\mathbf{z}} = \lim_{\nu \to \infty} \bar{\mathbf{z}}_\nu \in \Gamma_{\overline{A}} \setminus \{\mathbf{o}\}$. Then X_{11} must be bounded (and positive). Now $\bar{\mathbf{x}} = \frac{1}{X_{11}} \mathbf{x} \in \mathcal{T}$ must be bounded too, since $\mathcal{T}$ is compact. So $\mathbf{x} = X_{11} \bar{\mathbf{x}}$ must be bounded. □

Finally we sharpen the assumption on B to be positive-definite, to derive boundedness of the feasible region.

Corollary 3.1 *If B is positive-definite, then under the assumption (10) the feasible set of (21) is bounded.*

Proof. If $\overline{B} \bullet X = 1$, then $B \bullet Y = 1 - \beta X_{11} - 2\mathbf{b}^\top \mathbf{x}$ must be bounded by Lemma 5. Now choose $\rho > 0$ such that $B - \rho I \in \mathcal{P}_n$. Then $\rho Y_{jj} \leq \rho I \bullet Y \leq B \bullet Y$ must be bounded, and therefore Y, too, since $Y \in \mathcal{P}$ implies $|Y_{jk}| \leq \sqrt{Y_{jj} Y_{kk}}$ for all j, k. □

Note that all above results hold a fortiori for the higher-order relaxations $\mathcal{K}^r_{n+1} \subseteq \mathcal{D}_{n+1}$.

4 Standard Fractional Quadratic Problem

4.1 Formulation

The *Standard Fractional Quadratic Problem* (StFQP) is a CFQP where the constraint set is the standard simplex Δ as defined in (1). The StFQP is *NP*-hard as the StQP is also *NP*-hard. Despite the constraints being simpler, this problem class retains most of the complexity of the previous polyhedron case. It is possible to transform a bounded CFQP into an equivalent StFQP using a vertex based representation. This reduction is not useful in practice if the number of vertices is large, but in any case it helps to establish theoretical results. The importance of the StFQP is well established from the fact that it can be used to formulate some combinatorial optimization problems. Also in branch-and-bound methods for global fractional quadratic optimization, a simplex partition of the domain is often used, such that each node in the branch-and-bound tree corresponds to a StFQP.

Nonhomogeneous quadratic expressions $q(\mathbf{x}) = \mathbf{x}^\top \hat{C} \mathbf{x} + 2\mathbf{c}^\top \mathbf{x} + \gamma$ over the simplex Δ can be made homogeneous by defining $C = \hat{C} + \mathbf{c}\mathbf{e}^\top + \mathbf{e}\mathbf{c}^\top + \gamma \mathbf{e}\mathbf{e}^\top$ so that $\mathbf{x}^\top C \mathbf{x} = q(\mathbf{x})$ for all $\mathbf{x} \in \Delta$. So in this section we consider, without loss of generality, the problem

$$\min \left\{ \frac{\mathbf{x}^\top C \mathbf{x}}{\mathbf{x}^\top B \mathbf{x}} : \mathbf{x} \in \Delta \right\} . \tag{23}$$

In this context, Lemma 1 reduces to the evident fact that $\mathbf{x}^\top B \mathbf{x} > 0$ for all $\mathbf{x} \in \Delta$ if and only if B is strictly copositive. In turn, this condition is equivalent to our overall model assumption (10) in context of StFQP.

In the particular case of a StFQP, dual attainability was implicitly already established in [39, Theorem 3.5]:

$$\begin{aligned} \min\left\{\frac{\mathbf{x}^\top C\mathbf{x}}{\mathbf{x}^\top B\mathbf{x}} : \mathbf{x}\in\Delta\right\} &= \min\left\{C\bullet X : B\bullet X = 1\,,\; X\in\mathcal{C}_n^*\right\} \\ &= \max\left\{\lambda : C-\lambda B\in\mathcal{C}_n\right\}\,. \end{aligned} \tag{24}$$

In the paper [39], Preisig has developed a bisection method based upon the last reformulation, using a copositivity test as a black box. In view of recent developments in copositivity testing, see in particular [15, 10], it may be worth while to revisit this approach, but in this paper we follow a different one.

Comparing the resulting pair in (24) to the original CP formulation in (14) and in (17), we obtain a dimension reduction from $n+1$ as in the general CFQP case to n in the StFQP case. Based on this formulation, we proceed to lower bounds based on the SDP relaxation of (24). Let

$$\psi = \max\left\{\lambda : C-\lambda B\in\mathcal{C}_n\right\}\,.$$

As in the general description of Subsection 3.3, we again employ the cone of doubly nonnegative matrices $\mathcal{D}_n = (\mathcal{P}_n\cap\mathcal{N}_n)\supseteq\mathcal{C}_n$ with its dual cone $\mathcal{D}_n^* = \mathcal{P}_n+\mathcal{N}_n\subseteq\mathcal{C}_n^*$, and define, following [4] and [11],

$$\psi_{\text{cop}} = \max\left\{\lambda : C-\lambda B\in\mathcal{P}_n+\mathcal{N}_n\right\}\le\psi\,. \tag{25}$$

By strong duality

$$\begin{aligned} \psi_{\text{cop}} &= \min\left\{C\bullet X : B\bullet X = 1, X\in\mathcal{D}_n\right\} & (26)\\ &= \min\left\{C\bullet X : B\bullet X = 1, X\succeq 0, X\ge 0\right\}\,. & (27) \end{aligned}$$

Hence ψ_{cop} is a lower bound for (23). In analogy to (22), a stronger lower bound can be found by solving

$$\psi_{\text{cop}}^+ = \min\left\{C\bullet X : B\bullet X = 1, X\in\mathcal{P}_n,\; 0\le X\le E\right\}, \tag{28}$$

where $E = \mathbf{e}\mathbf{e}^\top$ is the $n\times n$ all-ones matrix. Here we use the fact that for all $\mathbf{x}\in\Delta$, we have $x_ix_j\le 1$, all i,j, so that $X = \mathbf{x}\mathbf{x}^\top\le E$. Therefore

$$\psi_{\text{cop}}\le\psi_{\text{cop}}^+\le\psi\,. \tag{29}$$

4.2 Application of StFQP: Symmetric eigenvalue complementarity problem

Given matrices $\left\{\widehat{A},\widehat{B}\right\}\subset\mathcal{M}$ with $\widehat{B}$ pd, the Symmetric Eigenvalue Complementarity Problem (EiCP) [40], [44] consists in finding

$$\lambda>0 \text{ and } \mathbf{x}\in\mathbb{R}^n_+\setminus\{\mathbf{o}\} \text{ such that } \mathbf{w} := (\lambda\widehat{B}-\widehat{A})\mathbf{x}\in\mathbb{R}^n_+ \quad\text{and } \mathbf{x}^\top\mathbf{w} = 0\,.$$

For any solution $(\lambda,\mathbf{x})$ of EiCP, the value of λ is called Complementary Eigenvalue of the matrices $(\widehat{A},\widehat{B})$ and $\mathbf{x}$ is the corresponding Complementary Eigenvector. The symmetric EiCP can be reduced to the problem of finding a stationary point of the Rayleigh function on the simplex [40], for which a number of efficient global nonlinear optimization algorithms can be useful [30]. This problem has found applications in the study of resonance frequency of structures and stability of dynamical systems [19]. In practice, it is important to find the maximum complementary eigenvalue for the EiCP. A sequential algorithm for this purpose has been introduced in [31]. Alternatively, such an eigenvalue can be computed as a global minimum of the StQFP (23) with $C = -\widehat{A}$ and $B = \widehat{B}$.

4.3 Inconsistent systems of linear constraints

The repair of an inconsistent system is an important application of the CFQP. Suppose that we are given a convex set $\mathbf{X} \subseteq \mathbb{R}^n$, an $m \times n$ matrix A, and a vector $\mathbf{a} \in \mathbb{R}^n$ which form a system of linear (in)equalities $A\mathbf{x} \left(\begin{smallmatrix} = \\ \leq \end{smallmatrix} \right) \mathbf{a}$ that has no solution $\mathbf{x} \in X$.

An interesting formulation of this inconsistent problem consists of minimizing the *Frobenius norm correction* $[H, \mathbf{p}]$ of the matrix $[A, \mathbf{a}]$, that is,

$$
\begin{array}{lll}
(P_I): \quad \phi = \min & \|[H, \mathbf{p}]\|_F^2 & \\
\text{subject to} & (A + H)\mathbf{x} \left(\begin{smallmatrix} = \\ \leq \end{smallmatrix} \right) \mathbf{a} + \mathbf{p} & \qquad (30) \\
& H \in \mathbb{R}^{m \times n}, \quad \mathbf{p} \in \mathbb{R}^m, \quad \mathbf{x} \in \mathbf{X}\,. &
\end{array}
$$

The interest in formulating this correction problem lies not only in a direct diagnosis and correction of the infeasible model, but also in an insight into the nature of the infeasibility, that is provided by the "near" feasible solution of problem (30).

Problem (30) was shown [1] to be equivalent to the following CFQP, where without loss of generality we assume that $\left(\begin{smallmatrix} = \\ \leq \end{smallmatrix} \right)$ represents $m-r$ initial equalities, followed by r inequalities.

$$
(\mathrm{P}_F): \quad \phi = \min \quad \frac{\|\mathbf{v}\|^2}{1 + \|\mathbf{x}\|^2} \tag{31}
$$

$$
\text{subject to} \quad A\mathbf{x} - \mathbf{v} \left(\begin{smallmatrix} = \\ \leq \end{smallmatrix} \right) \mathbf{a} \tag{32}
$$

$$
v_i \geq 0 \text{ for } i = m - r + 1, \cdots, m \tag{33}
$$

$$
\mathbf{x} \in \mathbf{X}\,. \tag{34}
$$

Suppose that $\mathbf{X} = \mathbb{R}^n_+$. Accordingly to the $m - r$ initial equalities, and r inequalities, let $A = \begin{bmatrix} A_{m-r} \\ A_r \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \mathbf{v}_{\overline{r}} \\ \mathbf{v}_r \end{bmatrix}$. Introducing a vector of r slack variables $\mathbf{s} \in \mathbb{R}^r_+$ in the inequality constraints, problem (P_F) is a particular case of problem (4) with

$$
C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{c} = \mathbf{o}, \gamma = 0, B = \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{b} = \mathbf{o}, \beta = 1
$$

and

$$\mathcal{T} = \left\{ \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \\ \mathbf{s} \end{bmatrix} \in \mathbb{R}^{n+m+r}_+ : \begin{bmatrix} A_{m-r} & -I_{m-r} & 0 & 0 \\ A_r & 0 & -I_r & I_r \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{v}_{m-r} \\ \mathbf{v}_r \\ \mathbf{s} \end{bmatrix} = \mathbf{a} \right\} .$$

Now consider a special case of (30) with no restrictions on $\mathbf{x}$ (so $X = \mathbb{R}^n$) and only equality constraints:

$$\begin{array}{ll} (\mathrm{P}_E) & \min \|[H, \mathbf{p}]\|_F^2 \\ \text{subject to} & (A+H)\mathbf{x} = \mathbf{a} + \mathbf{p} \\ & H \in \mathbb{R}^{m\times n}, \quad \mathbf{p} \in \mathbb{R}^m, \quad \mathbf{x} \in \mathbb{R}^n . \end{array} \tag{35}$$

This problem is relatively easy to solve, as it can be reduced to a Total Least Squares Problem (TLSP). If additional constraints, such as $\mathbf{x} \geq \mathbf{o}$ are introduced, then a more difficult problem has to be tackled. There are many applications of this problem, for instance, in regression analysis when the coefficients of the model must be non-negative, and noise is assumed both in the input as in the output data.

An unconstrained formulation for this problem exists [1], and is given by

$$(\mathrm{P}_{\mathrm{NL}0}) \qquad \phi = \inf \left\{ \frac{\|A\mathbf{x} - \mathbf{a}\|^2}{1 + \|\mathbf{x}\|^2} : \mathbf{x} \in \mathbb{R}^n_+ \right\} . \tag{36}$$

We can rephrase (36) as a homogeneous quadratic fractional problem

$$(\mathrm{P}_{\mathrm{NL}1}) \qquad \psi \;=\; \inf \left\{ g(\mathbf{z}) : \mathbf{z} \in \mathbb{R}^{n+1}_+ ,\ z_1 > 0 \right\} \tag{37}$$

where $g(\mathbf{z}) = \frac{\mathbf{z}^\top \overline{A}\mathbf{z}}{\mathbf{z}^T\mathbf{z}}$, and, as before,

$$\overline{A} = [-\mathbf{a} \quad A]^T [-\mathbf{a} \quad A] = \begin{bmatrix} \mathbf{a}^\top\mathbf{a} & -\mathbf{a}^\top A \\ -A^\top\mathbf{a} & A^\top A \end{bmatrix} . \tag{38}$$

Note that $\overline{A}$ replaces C in the general StFQP formulation (23), and that $\overline{A}$ plays a different role in the general CFQP formulation.

We use (37) and introduce some results that, under a sufficient condition easily verifiable, allows to drop constraint $z_1 > 0$ in favor of the more manageable constraint $z_1 \geq 0$. Under the same assumptions, we prove that (36) is equivalent to a StFQP.

Theorem 4 *Let*

$$(\mathrm{P}_{\mathrm{NL}2}) \quad \psi = \min \{ g(\mathbf{z}) : \mathbf{z} \in \Delta \} , \tag{39}$$

and

$$(\mathrm{P}_{\mathrm{NL}3}) \quad \varsigma = \inf \left\{ g(\mathbf{z}) : \mathbf{z} \in \mathbb{R}^n_+ \right\} . \tag{40}$$

Then $\psi = \varsigma$.

Proof. The existence of an optimal solution of $(\mathrm{P}_{\mathrm{NL}2})$ is obvious. By inclusion we know that $\varsigma \leq \psi$. Now suppose that $\varsigma < \psi$. Then there exists a vector $\mathbf{z}_0 \in \mathbb{R}^n_+ \setminus \{\mathbf{o}\}$ and an optimal solution $\mathbf{z}_1 \in \Delta$ of (39) such that

$$\frac{\mathbf{z}_0^\top \overline{A}\mathbf{z}_0}{\mathbf{z}_0^\top \mathbf{z}_0} < \frac{\mathbf{z}_1^T \overline{A}\mathbf{z}_1}{\mathbf{z}_1^\top \mathbf{z}_1} \,.$$

Since $\frac{\mathbf{z}_0}{\mathbf{z}_0^\top \mathbf{e}}$ is also a feasible solution of (39), then $\mathbf{z}_1$ cannot be an optimal solution of (39). Hence $\psi = \varsigma$. □

Theorem 5 *Let* $\overline{\mathbf{x}}$ *be a global solution to*

$$\min \left\{ \frac{\mathbf{x}^\top A^\top A\mathbf{x}}{\mathbf{x}^\top \mathbf{x}} : \mathbf{x} \in \Delta \right\} \,. \tag{41}$$

If

$$(A^\top \mathbf{a})^\top \overline{\mathbf{x}} > 0 \,, \tag{42}$$

then

$$(\mathrm{P_{NL1}}) : \inf \left\{ g(\mathbf{z}) : z_1 > 0 \,, \mathbf{z} \in \mathbb{R}^n_+ \right\} \tag{43}$$

is equivalent to

$$(\mathrm{P_{NL2}}) : \min \left\{ g(\mathbf{z}) : \mathbf{z} \in \Delta \right\} \,. \tag{44}$$

Proof. By Theorem 4, $(\mathrm{P_{NL2}})$ is equivalent to $(\mathrm{P_{NL3}})$, so it is sufficient to show that $\overline{\mathbf{z}} = \begin{pmatrix} 0 \\ \overline{\mathbf{x}} \end{pmatrix}$ cannot be an optimal solution of $(\mathrm{P_{NL2}})$. Supposing the contrary, then $\overline{\mathbf{z}} = \begin{pmatrix} 0 \\ \overline{\mathbf{x}} \end{pmatrix}$ satisfies the KKT conditions

$$\begin{array}{rcl} \nabla g(\mathbf{z}) & = & \lambda \mathbf{e} + \mathbf{w} \\ \mathbf{w} \geq \mathbf{o} & , & \mathbf{z} \geq \mathbf{o} \\ \mathbf{z}^\top \mathbf{w} & = & 0 \\ \mathbf{e}^\top \mathbf{z} & = & 1 \end{array}$$

where $\nabla g(\mathbf{z})$ represents the gradient of g at $\mathbf{z}$. But

$$\nabla g(\mathbf{z}) = \frac{2}{\mathbf{z}^\top \mathbf{z}} \left[\overline{A}\mathbf{z} - \mu \mathbf{z} \right] \,,$$

where $\mu = g(\mathbf{z})$. Furthermore, by Euler's homogeneity theorem,

$$0 = \overline{\mathbf{z}}^\top \nabla g(\overline{\mathbf{z}}) = \lambda \mathbf{e}^\top \overline{\mathbf{z}} + 0 = \lambda \,.$$

Then $\overline{\mathbf{z}} = \begin{pmatrix} 0 \\ \overline{\mathbf{x}} \end{pmatrix}$ satisfies

$$\overline{A}\overline{\mathbf{z}} - \mu \overline{\mathbf{z}} = 2(\overline{\mathbf{z}}^\top \overline{\mathbf{z}}) \nabla g(\overline{\mathbf{z}}) = 2(\overline{\mathbf{z}}^\top \overline{\mathbf{z}})\mathbf{w} \geq \mathbf{o} \,,$$

that is

$$\begin{bmatrix} \mathbf{a}^\top \mathbf{a} & -\mathbf{a}^\top A \\ -A^\top \mathbf{a} & A^\top A \end{bmatrix} \begin{bmatrix} 0 \\ \overline{\mathbf{x}} \end{bmatrix} - \mu \begin{bmatrix} 0 \\ \overline{\mathbf{x}} \end{bmatrix} \geq 0 \,.$$

Therefore, $-(A^\top \mathbf{a})^\top \overline{\mathbf{x}} \geq 0$, which is impossible by hypothesis. □

In practice, the following condition

$$\min_j [A^\top \mathbf{a}]_j > 0 \tag{45}$$

is sufficient for the equivalence of problems ($\mathrm{P}_{\mathrm{NL2}}$) and ($\mathrm{P}_{\mathrm{NL3}}$). In fact, as $\overline{\mathbf{x}} \in \Delta$, then

$$(A^\top \mathbf{a})^\top \overline{\mathbf{x}} \geq \min_j [A^\top \mathbf{a}]_j \sum_{i=1}^n \overline{x}_i = \min_j [A^\top \mathbf{a}]_j > 0\,.$$

Although more restrictive, this condition (45) is easily verifiable.

5 Computational experience

In this section we report encouraging numerical experience for a set of randomly generated CFQPs. Lower bounds obtained by the SDP relaxation of the completely positive conic formulation are presented. To solve the SDP problems, the self-dual SDP code `SeDuMi` [46] was used, with the interface code `YALMIP` [34].

These values were compared with the lower bound obtained by `Gloptipoly 3` [28], a software for the Generalized Problem of Moments (GPM) [32]. Any rational polynomial optimization problem over a semialgebraic set can be formulated as a linear moment problem [29]. `Gloptipoly 3` allows to build up a hierarchy of SDP relaxations of the GPM, to generate a monotonic sequence of optimal values converging to the global optimum.

In addition, we present a comparison with the lower bound at the root node, obtained by the well-known and robust global optimization code `BARON` (Branch And Reduce Optimization Navigator) [43], which combines constraint propagation, interval analysis, and duality in an enhanced branch-and-reduce framework. The optimal value obtained by `BARON` was used to establish the gaps of the lower bounds.

When generating instances of program (19), some specific remarks seem to be in order. A naive direct implementation of the copositive relaxation (21) introduces numerical difficulties when solving the SDP problem, due to the homogeneous constraint $\overline{A} \bullet X = 0$; c.f. [3], where it is mentioned that the *"SDP may be unbounded even though all of the original variables have finite upper and lower bounds"* (albeit for a possibly indefinite $\overline{A}$ there); note that the latter difficulty is excluded under additional assumptions, as shown in Corollary 3.1.

Here we propose a simple transformation which even results in immediate size reduction, basically from n^2 to $(n-m)^2$. So let $\overline{A}$ be psd but singular, so that $\dim\ker \overline{A} = k+1$ for some $k \in \mathbb{N}$. First we describe an orthonormal basis of this kernel. Remember that A is supposed to be an $m \times n$ matrix with full row rank $m < n$ (to allow for a non-trivial feasible set $\mathcal{T}$). Then $A^\top A$ is psd but has a kernel of dimension $k = n - m$, spanned by the orthonormal vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$, say. So the $m \times m$ matrix $AA^\top$ is nonsingular, and it is easy to see that $\overline{\mathbf{u}}_i = [0, \mathbf{u}_i^\top]^\top$ form an orthonormal system in $\ker \overline{A} \subseteq \mathbb{R}^{n+1}$, as detailed, e.g., in the proof of Lemma 4. Next, denote by

$$\tilde{\mathbf{u}}_0 = \begin{bmatrix} 1 \\ A^\top (AA^\top)^{-1}\mathbf{a} \end{bmatrix} \quad \text{and} \quad \overline{\mathbf{u}}_0 = \frac{1}{\|\tilde{\mathbf{u}}_0\|}\,\tilde{\mathbf{u}}_0\,.$$

Then the orthonormal system $\{\overline{\mathbf{u}}_0, \ldots, \overline{\mathbf{u}}_k\}$ spans $\ker \overline{A}$, as can be checked in a straightforward manner, using again arguments from the proof of Lemma 4.

Now let Q be a $(k+1)\times(n+1)$ matrix, collecting the above system as rows: $Q^\top = [\overline{\mathbf{u}}_0, \ldots, \overline{\mathbf{u}}_k]$. It follows that, for any $X \in \mathcal{D}_{n+1}$, we have $\overline{A} \bullet X = 0$ if and only if $\overline{A}X = O$ if and only if

$$X = Q^\top Y Q \quad \text{for some } Y \in \mathcal{P}_{k+1} \text{ satisfying } Q^\top Y Q \in \mathcal{N}_{n+1}\,. \tag{46}$$

Hence, using $\overline{C} \bullet X = (Q\overline{C}Q^\top) \bullet Y$ etc., we arrive at the reduced SDP

$$\min\left\{(Q\overline{C}Q^\top) \bullet Y : (Q\overline{B}Q^\top) \bullet Y = 1\,,\; Q^\top Y Q \geq O\,, Y \in \mathcal{P}_{k+1}\right\}, \tag{47}$$

working on smaller psd matrices, but retaining $\mathcal{O}(n^2)$ linear inequalities.

For $\beta = \gamma = 1$ and for selected values of n and $m = \lfloor \frac{n}{2} \rfloor$, we have generated instances of program (4) as follows:

1. a symmetric psd $n \times n$ matrix B is randomly generated, along with a suitably scaled vector $\mathbf{b} \in \operatorname{int}(\mathbb{R}^n_+)$ such that $\overline{B}$ given by (7) need not be psd, and can have negative entries (but obviously $\overline{B} \in \mathcal{D}^*_{n+1}$). Observe that by construction, $\overline{B}$ is strictly $\mathbb{R}^n_+$-copositive and therefore, for any choice of $\overline{A}$, strictly $\Gamma_{\overline{A}}$-copositive for sure.
2. a (possibly indefinite) symmetric $n\times n$ matrix C is randomly generated with entries of varying sign, along with a randomly drawn vector $\mathbf{c} \in \mathbb{R}^n$ (again, no sign restrictions on the coordinates).
3. an $m \times n$ matrix A with a strictly positive first row, but varying sign of entries elsewhere, is randomly generated;
4. an arbitrary vector $\mathbf{x} \in \Delta$ is drawn at random. Then the choice $\mathbf{a} = A\mathbf{x}$ ensures that $\mathcal{T}$ is compact, so the model assumptions (10) are guaranteed.
5. Finally, based on $(A, \mathbf{a})$, the matrix Q is determined and a solution Y to (47) is calculated. As stated before, $X = Q^\top Y Q$ solves (21). The objective $\overline{C} \bullet X = (Q\overline{C}Q^\top) \bullet Y$ is used as a relaxation bound.

Instances of sizes $n \in \{4, 9, 49, 79\}$ were generated, resulting in SDP instances of dimensionality 5, 10, 50 and 80, which numbers appear in the instance name as the first number after ABJ. The nonnegativity constraint $X \geq O$ pose a notorious impediment on the problem size to allow for satisfactory handling by any SDP solver. The maximum size of 80 was possible due to the size reduction achieved in (47). The clear impact of this reduction is depicted in the three last columns of Figure 1. For two problems of size 5 (ABJ5_0) and 10 (ABJ10_0), `SeDuMi` output reports the size of the SDP problems, for the GPM approach, the direct copositive relaxation (21) and with (47).

Table 1 reports for each instance the information,

	GPM (Gloptipoly3)	Copositive Relaxation	Copositve Relaxation with Reduction
ABJ5_0	eqs m = 210, order n = 98, dim = 2380, blocks = 7 nnz(A) = 2385 + 0, nnz(ADA) = 44100, nnz(L) = 22155 Detailed timing (sec) Pre IPM Post 7.001E-03 3.920E-01 2.002E-03	eqs m = 15, order n = 33, dim = 53, blocks = 3 nnz(A) = 55 + 0, nnz(ADA) = 225, nnz(L) = 120 Detailed timing (sec) Pre IPM Post 5.200E-02 7.001E-02 9.958E-04	eqs m = 6, order n = 27, dim = 33, blocks = 3 nnz(A) = 121 + 0, nnz(ADA) = 36, nnz(L) = 21 Detailed timing (sec) Pre IPM Post 4.003E-03 3.800E-02 9.958E-04
ABJ10_0	eqs m = 5005, order n = 718, dim = 85638, blocks = 12 nnz(A) = 138325 + 0, nnz(ADA) = 25050025, nnz(L) = 12527515 Detailed timing (sec) Pre IPM Post 8.460E-01 4.794E+02 2.800E-02	eqs m = 55, order n = 113, dim = 203, blocks = 3 nnz(A) = 210 + 0, nnz(ADA) = 3025, nnz(L) = 1540 Detailed timing (sec) Pre IPM Post 6.100E-02 6.500E-02 1.006E-03	eqs m = 15, order n = 99, dim = 119, blocks = 3 nnz(A) = 1291 + 0, nnz(ADA) = 225, nnz(L) = 120 Detailed timing (sec) Pre IPM Post 2.900E-02 5.301E-02 1.992E-03

Figure 1: Sizes of SDP relaxations

- Instance — Instance name;
- Cop R — Value of the lower bound obtained by the SDP relaxation of the copositive formulation (47);
- Time1(s) — CPU time in seconds to obtain Cop R;
- Gap — The relative gap provided by Cop R,

$$\left| \frac{\text{Cop R-}\texttt{BARON}\text{ Optimal value}}{\texttt{BARON}\text{ Optimal value}} \right| ;$$

- GPM — Value of the lower bound obtained by `Gloptipoly 3`;
- Time2(s) — CPU time in seconds to obtain the GPM lower bound;
- St — Status of `Gloptipoly 3` solution for the default relaxation order;
- root B — Value of the lower bound obtained at the root node by `BARON`;

All the tests have been performed on a Pentium Intel(R) Core(TM)i7, with CPU E8400, 2.8GHz, 4,00 GB RAM, and 64-bit operating system Windows. A tolerance parameter 10^{-4} was considered for `BARON` and `SeDuMi`.

An analysis of Table 1 reveals that the lower bounds provided by solving the SDP relaxation of the Copositive formulation are very good, as the gaps show, and outperforms the initial lower bound of `BARON`, and of the GPM relaxation. For problems of size 50 and 80 `Gloptipoly`

Table 1: Copositive Relaxation versus `Gloptipoly 3` and `BARON`

Instance	Cop R	Time1(s)	Gap	GPM R	Time2(s)	St.	root B.
ABJ5_0	-0.7865	2.700e-02	0.4837	-0.5275	1.045e+00	1	-26.1028
ABJ5_1	-0.4923	3.400e-02	1.8293	-0.5414	1.014e+00	1	-11.8308
ABJ5_2	-0.7693	2.700e-02	0.6771	-0.5089	9.672e-01	1	-11.9631
ABJ5_3	-0.3603	2.900e-02	0.9907	-0.2207	1.310e+00	1	-3.9613
ABJ5_4	-1.2562	2.700e-02	0.5467	-0.9428	9.984e-01	1	-0.8123
ABJ5_5	+0.4643	3.000e-02	0.1552	+0.2225	1.108e+00	1	-2.2940
ABJ5_6	-0.5768	3.100e-02	0.5831	-0.3671	9.828e-01	1	-8.6291
ABJ5_7	-0.0815	3.300e-02	15.2108	-0.0657	8.892e-01	1	-5.1034
ABJ5_8	-0.5946	2.600e-02	0.4752	-0.3708	9.516e-01	1	-0.4031
ABJ5_9	-0.8705	3.100e-02	0.9123	-0.5753	6.708e-01	1	-0.4553
ABJ10_0	-0.3095	3.500e-02	0.5090	-0.1962	7.010e+02	1	-23.9325
ABJ10_1	-0.6779	3.100e-02	0.4781	-0.4882	5.737e+02	1	-0.4587
ABJ10_2	+0.4144	3.400e-02	0.0533	+0.4288	6.395e+02	1	-3.4076
ABJ10_3	-0.3105	3.500e-02	1.2843	-0.1840	6.298e+02	1	-12.3357
ABJ10_4	-0.3885	3.900e-02	0.4746	-0.2689	5.122e+02	1	-0.2635
ABJ10_5	-0.7710	4.300e-02	0.2028	-0.6198	6.619e+02	1	-55.5414
ABJ10_6	-1.2861	3.100e-02	0.5562	-0.8749	7.123e+02	1	-0.8265
ABJ10_7	-0.1154	3.900e-02	1.1720	-0.0760	6.219e+02	1	-25.4559
ABJ10_8	-0.6486	3.100e-02	0.2828	-0.4558	6.239e+02	1	-0.5056
ABJ10_9	-0.3070	4.800e-02	0.5997	-0.1794	6.183e+02	1	-0.1919
ABJ50_0	-0.7435	3.238e+00	0.3552	O of M	-		-502.4740
ABJ50_1	-0.9606	2.731e+00	0.2229	O of M	-		-0.7856
ABJ50_2	-0.7844	3.192e+00	0.2786	O of M	-		-0.6135
ABJ50_3	-0.4022	2.983e+00	0.3630	O of M	-		-1463.1800
ABJ50_4	-0.2677	3.001e+00	0.8199	O of M	-		-451.7790
ABJ50_5	-0.6484	2.981e+00	0.6369	O of M	-		-0.3962
ABJ50_6	-0.5760	3.498e+00	0.3702	O of M	-		-989.5200
ABJ50_7	-0.6486	2.993e+00	0.3201	O of M	-		-0.4914
ABJ50_8	-0.5985	3.221e+00	0.3456	O of M	-		-490.0360
ABJ50_9	-0.3730	3.244e+00	0.3215	O of M	-		-626.8870
ABJ80_0	-0.4427	5.049e+01	0.5019	O of M	-		-1394.8500
ABJ80_1	-0.5806	5.532e+01	0.2984	O of M	-		-0.4472
ABJ80_2	-0.8597	5.532e+01	0.2869	O of M	-		-0.6681
ABJ80_3	-0.4345	5.519e+01	0.3302	O of M	-		-1849.5000
ABJ80_4	-0.8625	5.101e+01	0.3214	O of M	-		-0.6528
ABJ80_5	-0.4670	5.117e+01	0.3301	O of M	-		-0.3511
ABJ80_6	-0.3473	5.539e+01	0.6090	O of M	-		-2488.4700
ABJ80_7	-0.5883	5.105e+01	0.3607	O of M	-		-1487.1000
ABJ80_8	-0.4181	5.532e+01	0.5004	O of M	-		-736.0130
ABJ80_9	-0.7023	5.099e+01	0.3568	O of M	-		-0.5177

`3` ran out of memory (O of M), as expected given the size of the corresponding SDP problem (as the results in Figure 1 suggested). As the status for `Gloptipoly 3` indicates, increasing the relaxation order has no effect. Moreover, the results show that the reduction proposed in (47) is crucial as the size of the problem increases.

In our opinion, the numerical results show that the SDP ideas discussed in this paper are promising to be incorporated in a robust branch-and-bound algorithm for dealing with the CFQP.

6 Conclusions

In this paper we present copositive exact formulations for the CFQP and the StFQP. The practical interest in these problems is discussed, with emphasis on the eigenvalue complementarity problem and the correction of inconsistent linear systems. For the StFQP we proved that dual attainability holds, while a more specific copositivity condition is needed for this result to hold for a general CFQP. Based on these formulations SDP relaxations are proposed providing good lower bounds. Theoretical results presented in this paper have important implications in the computation of lower bounds for the CFQP. Computational experience with SDP relaxation of the CFQP is presented showing small relative gaps. When compared with the initial lower bound given by BARON and Gloptipoly 3 the SDP relaxation of the copositive formulation produces better lower bounds, particularly when the size of problems increases. These SDP-based lower bounds seem useful to be included in a branch-and-bound approach to be developed in the future.

Acknowledgements. The authors are indebted to two anonymous referees and an anonymous Associate Editor for valuable remarks which led to several improvements of a previous version; for instance, Subsection 2.2 was added upon the suggestions of a referee.

References

[1] P. Amaral and P. Barahona. Connections between the total least squares and the correction of an infeasible system of linear inequalities. *Linear Algebra and Applications*, 395:191–210, 2005.

[2] P. Amaral, J. Júdice, and H. D. Sherali. A reformulation–linearization–convexification algorithm for optimal correction of an inconsistent system of linear constraints. *Computers and Operations Research*, 35:1494–1509, 2008.

[3] K. M. Anstreicher. Semidefinite programming versus the reformulation-linearization technique for nonconvex quadratically constrained quadratic programming. *Journal of Global Optimization*, 43:471–484, 2009.

[4] K. M. Anstreicher and S. Burer. D.C. versus copositive bounds for Standard QP. *Journal of Global Optimization*, 33:299–312, 2005.

[5] K M. Anstreicher and S. Burer. Computable representations for convex hulls of low-dimensional quadratic forms. *Mathematical Programming*, 124(1-2, Ser. B):33–43, 2010.

[6] C. Audet, P. Hansen, B. Jaumard, and G. Savard. A branch and cut algorithm for nonconvex quadratically constrained quadratic programming. *Mathematical Programming*, 87:131–152, 2000.

[7] A. Beck, A. Ben-Tal, and M. Teboulle. Finding global optimal solution for a quadratically constrained fractional quadratic problem with applications to the regularized total least squares. *SIAM Journal on Matrix Analysis and Applications*, 28:425–445, 2006.

[8] A. Beck and M. Teboulle. A convex optimization approach for minimizing the ratio of indefinite quadratic functions over an ellipsoid. *Mathematical Programming*, 118(1):13–35, 2009.

[9] I.M. Bomze, M. Dür, E. De Klerk, C. Roos, A.J. Quist, and T. Terlaky. On copositive programming and standard quadratic optimization problems. *Journal of Global Optimization*, 18:301–320, 2000.

[10] I.M. Bomze and G. Eichfelder. Copositivity detection by difference-of-convex decomposition and ω-subdivision. newblock *Mathematical Programming*, to appear.

[11] I.M. Bomze, M. Locatelli, and F. Tardella. New and old bounds for standard quadratic optimization: dominance, equivalence and incomparability. *Mathematical Programming*, 115:31–64, 2008.

[12] I.M. Bomze. Copositive optimization – recent developments and applications. *European Journal of Operational Research*, 216:509–520, 2012.

[13] I.M. Bomze and E. de Klerk. Solving standard quadratic optimization problems via linear, semidefinite and copositive programming. *Journal of Global Optimization*, 24(2):163–185, 2002.

[14] I.M. Bomze, W. Schachinger, and G. Uchida. Think co(mpletely)positive ! – matrix properties, examples and a clustered bibliography on copositive optimization. *Journal of Global Optimization*, 52:423–445, 2012.

[15] S. Bundfuss and M. Dür. Algorithmic copositivity detection by simplicial partition. *Linear Algebra and its Applications*, 428:1511–1523, 2008.

[16] S. Bundfuss and M. Dür. An adaptive linear approximation algorithm for copositive programs. *SIAM Journal on Optimization*, 20:30–53, 2009.

[17] S. Burer. On the copositive representation of binary and continuous nonconvex quadratic programs. *Mathematical Programming*, 120:479–495, 2009.

[18] S. Burer. Copositive programming. In M.F. Anjos and J.B. Lasserre, editors, *Handbook of Semidefinite, Cone and Polynomial Optimization: Theory, Algorithms, Software and Applications.* 201–218, International Series in Operations Research and Management Science. Springer, New York, 2012.

[19] A.P. Costa, I.N. Figueiredo, J. Júdice, and J.A.C. Martins. The directional instability problem in systems with frictional contacts. *Computer Methods in Applied Mechanics and Engineering*, 193:357384, 2004.

[20] P.J.C. Dickinson and L. Gijben. On the computational complexity of membership problems for the completely positive cone and its dual. Technical Report, Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, The Netherlands, 2011.

[21] H. Dong. Symmetric tensor approximation hierarchies for the completely positive cone. Working paper,, Dept. of Management Sciences, University of Iowa, Iowa City, IA, http://www.optimization-online.org/DB_HTML/2010/11/2791.html, 2010.

[22] M. Dür. Copositive programming — a survey. In M. Diehl, F. Glineur, E. Jarlebring, and W. Michiels, editors, *Recent Advances in Optimization and its Applications in Engineering*, pages 3–20. Springer, Berlin Heidelberg New York, 2010.

[23] G. H. Golub, P. C. Hansen, and D. P. O'Leary. Tikhonov regularization and total least squares. *SIAM Journal of Matrix Analysis and Applications*, 21:185–194, 1999.

[24] G. H. Golub and C. F. Van Loan. *Matrix Computations*. Johns Hopkins University Press, 3rd edition, 1996.

[25] J-Y. Gotoh and H. Konno. Maximization of the ratio of two convex quadratic functions over a polytope. *Computational Optimization and Applications*, 20(1):43–60, 2001.

[26] N. Gvozdenovic and M. Laurent. The operator Ψ for the chromatic number of a graph. *SIAM Journal on Optimization*, 19(2):572–591, 2008.

[27] M. Hall, Jr. and M. Newman. Copositive and completely positive quadratic forms. *Proc. Cambridge Philos. Soc.*, 59:329–339, 1963.

[28] D. Henrion, J.B. Lasserre, and J. Löfberg. Gloptipoly 3: moments, optimization and semidefinite programming. *Optimization Methods and Software*, 24:761–779, 2009.

[29] D. Jibetean and E. de Klerk. Global optimization of rational functions: A semidefinite programming approach. *Mathematical Programming*, 106(1, Ser. A):93–109, 2006.

[30] J. Júdice, S. Raydan, S. Rosa, and S. Santos. On the solution of the symetric eigenvalue complementarity problem by the spectral projected gradient algorithm. *Numerical Algorithms*, 47:391–407, 2008.

[31] J. Júdice, H. D. Sherali, I. M. Ribeiro, and S. Rosa. On the asymetric eigenvalue complementarity problem. *Optimization Methods and Software*, 24:549–568, 2009.

[32] J.B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM Journal on Optimization*, 11(3):796–817, 2001.

[33] J. Linderoth. A simplicial branch-and-bound algorithm for solving quadratically constrained quadratic programs. *Mathematical Programming*, 103:251–282, 2005.

[34] J. Löfberg. Yalmip : A toolbox for modeling and optimization in MATLAB. In *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.

[35] K.G. Murty and S.N. Kabadi. Some NP-complete problems in quadratic and nonlinear programming. *Mathematical Programmming*, 39(2):117–129, 1987.

[36] P.A. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD thesis, California Institute of Technology, Pasadena, CA, May 2000.

[37] P.A. Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programmming*, 96(2, Ser. B):293–320, 2003.

[38] J. Peña, J. Vera, and J.F. Zuluaga. Computing the stability number of a graph via linear and semidefinite programming. *SIAM Journal on Optimization*, 18(1):87–105, 2007.

[39] J.C. Preisig. Copositivity and the minimization of quadratic functions with nonnegativity and quadratic equality constraints. *SIAM Journal on Control and Optimization*, 34(4):1135–1150, 1996.

[40] M.G. Queiroz, J.J. Júdice, and C. Humes, Jr. The symmetric eigenvalue complementarity problem. *Mathematics of Computation*, 73:1849–1863, 2004.

[41] R. Renaut and A.H. Guo. A regularized total least squares algorithm. In S. Van Huffel and P. Lemmerling, editors, *Total Least Squares and Errors-in-Variables Modeling: Analysis, Algorithms and Applications*, pages 57–66. Kluwer Academic, 2002.

[42] R. Renaut and A.H. Guo. Efficient algorithms for solution of regularized total least squares. *SIAM Journal on Matrix Analysis and Applications*, 26(2):457–476, 2005.

[43] N. Sahinidis and M. Tawarmalani. *BARON: The GAMS Solver Manual. GAMS Development Corporation.* Washington, DC, 2004.

[44] A. Seeger. Eigenvalue analysis of equilibrium processes defined by linear complemetarity condition. *Linear Algebra and Applications*, 292:1–14, 1999.

[45] D.M. Sima, S. Van Huffel, and G.H. Golub. Regularized total least squares based on quadratic eigenvalue problem solvers. *Bit Numerical Mathematics*, 44(4):793–812, 2004.

[46] J.F. Sturm. Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11-12:625–653, 1999.

[47] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38:49–95, 1996.

[48] R. Yamamoto and H. Konno. An efficient algorithm for solving convex-convex quadratic fractional programs. *Journal of Optimization Theory and Applications*, 133(2):241–255, 2007.

[49] E.A. Yıldırım. On the accuracy of uniform polyhedral approximations of the copositive cone. *Optimization Methods and Software* 27(1):155–173, 2012.