

On the Quadratic Eigenvalue Complementarity Problem

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Abstract

We introduce several new results on the Quadratic Eigenvalue Complementarity Problem (QEiCP), focusing on the nonsymmetric case, i.e, without making symmetry assumptions on the matrices defining the problem. First we establish a new sufficient condition for existence of solutions of this problem, which is somewhat more manageable than previously existent ones. This condition works through the introduction of auxiliary variables which leads to the reduction of QEiCP to an Eigenvalue Complementarity Problem (EiCP) in higher dimension. Hence, this reduction suggests a new strategy for solving QEiCP, which is also analyzed in the paper. We also present an upper bound for the number of solutions of QEiCP and exhibit some examples of instances of QEiCP whose solution set has large cardinality, without attaining though the just mentioned upper bound. We also investigate the numerical solution of the QEiCP by exploiting a nonlinear programming and a variational inequality formulations of QEiCP. Some numerical experiments are reported and illustrate the benefits and drawbacks of using these formulations for solving the QEiCP in practice.

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1 Introduction

Given matrices $B, C \in \mathbb{R}^{n \times n}$, the *Eigenvalue Complementarity Problem* (to be denoted $\text{EiCP}(B, C)$, see e.g. [19] and [20]), consists of finding $(\lambda, x, w) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$w = \lambda Bx - Cx \tag{1}$$

$$w \geq 0, x \geq 0 \tag{2}$$

$$x^t w = 0 \tag{3}$$

$$e^t x = 1, \tag{4}$$

with $e = (1, 1, \dots, 1)^t \in \mathbb{R}^n$. The last constraint has been introduced, without loss of generality, in order to prevent the x component of a solution to vanish. Usually, the matrix B is assumed to be positive definite. In this paper, our basic assumption is the strict copositivity (SC) of B (see Definition 1(i) in Section 2, and note that PD matrices are SC). The problem has many applications in engineering (see [17], [20]), and can be seen as a generalization of the well-known Generalized Eigenvalue Problem, denoted GEiP (see e.g. [9]). Indeed, GEiP consists of solving just (1) with $w = 0$, and a solution (λ, x) of GEiP is just an eigenvalue and eigenvector of the matrix $B^{-1}C$ in the usual sense, when B is nonsingular. If a triplet (λ, x, w) solves EiCP, then the scalar λ is called a *complementary eigenvalue* and x is a *complementary eigenvector* associated to λ for the pair (B, C) . The condition $x^t w = 0$ and the nonnegative requirements on x and w imply that either $x_i = 0$ or $w_i = 0$ for $1 \leq i \leq n$. These two variables are called complementary.

It is easy to prove that under strict copositivity of B , $\text{EiCP}(B, C)$ always has a solution, because it can be reformulated as the Variational Inequality Problem $\text{VIP}(\bar{F}, \Omega)$ with feasible set $\Omega = \{x \in \mathbb{R}^n : e^t x = 1, x \geq 0\}$ and operator $\bar{F} : \Omega \rightarrow \mathbb{R}^n$ given by

$$\bar{F}(x) = \frac{x^t C x}{x^t B x} Bx - Cx,$$

see [13]. Note that \bar{F} is continuous in Ω as a consequence of the strict copositivity of B , and that Ω is convex and compact. It is well known that these two conditions ensure existence of solutions of $\text{VIP}(\bar{F}, \Omega)$ (see e.g. [4]). The reformulation of EiCP as a variational inequality problem is further developed in Section 3.

If the matrices B and C are both symmetric, then EiCP is called symmetric and reduces to the problem of finding a *Stationary Point* (SP) of the so-called Rayleigh Quotient function on the simplex Ω (see, e.g. [19, 20]), which is just a SP of the following Standard Quadratic Fractional Program

$$\begin{aligned} & \text{Maximize} && \frac{x^t C x}{x^t B x} \\ & \text{subject to} && e^t x = 1 \\ & && x \geq 0. \end{aligned}$$

A number of techniques have been proposed for solving the EiCP and its extensions, see e.g. [1], [2], [6], [7], [11], [12], [13], [14], [16], [18] and [22]. As expected, the symmetric EiCP is easier to solve.

Recently an extension of the EiCP has been introduced in [21], where some applications are highlighted. It has been named *Quadratic Eigenvalue Complementarity Problem* (QEiCP), and it differs from EiCP through the existence of an additional quadratic term on λ . Its formal definition follows.

Given $A, B, C \in \mathbb{R}^{n \times n}$, QEiCP(A, B, C) consists of finding $(\lambda, x, w) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$w = \lambda^2 Ax + \lambda Bx + Cx, \tag{5}$$

$$w \geq 0, x \geq 0, \tag{6}$$

$$x^t w = 0, \tag{7}$$

$$e^t x = 1, \tag{8}$$

where, as before, $e = (1, 1, \dots, 1)^t \in \mathbb{R}^n$. As in the case of the EiCP, (8) has been introduced, without loss of generality, for preventing the x component of a solution of the problem from vanishing. Note that when $A = 0$ QEiCP(A, B, C) reduces to EiCP($B, -C$). The λ component of a solution of QEiCP(A, B, C) is called a *quadratic complementary eigenvalue* for A, B, C , and the x component a *quadratic complementary eigenvector* for A, B, C associated to λ .

The case of the symmetric QEiCP, i.e., when A, B and C are symmetric matrices, has been fully analyzed in [5], where each instance of QEiCP with $n \times n$ matrices is related to an instance of EiCP with $2n \times 2n$ matrices. In this paper we remove such symmetry assumption, and focus on the general QEiCP. We also propose a connection between an n -dimensional QEiCP and a higher dimensional EiCP, but our connection procedure, developed in Section 2, differs from the one in [5].

We start by discussing the issue of existence of solutions. Contrary to the EiCP, the QEiCP may lack solutions, even under strict copositivity or positive definiteness of A . Indeed if we consider QEiCP($I, 0, I$), then premultiplying (6) by x and using (7), one gets $0 = (\lambda^2 + 1) \|x\|^2$, which has no solution $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n$ because (8) implies that $x \neq 0$. In fact, this difference between EiCP and QEiCP in terms of existence of solutions mirrors the elementary fact that linear equations in one real variable always have solutions, while quadratic equations may fail to have them.

Thus, the issue of conditions on (A, B, C) ensuring existence of solutions of QEiCP(A, B, C) is a relevant one. The so-called co-regularity and co-hyperbolicity properties were introduced by A. Seeger in [21] as sufficient conditions for the existence of solutions of the QEiCP. In Section 2 we will present another sufficient condition, which neither implies nor is implied by Seeger's conditions, and discuss the relation between Seeger's conditions and ours.

Both Seeger's and our sufficiency proofs work through the reduction of QEiCP to two different variational inequality problems, and hence each condition suggest a strategy for solving QEiCP, which are discussed in Section 3.

An upper bound for the number of complementary eigenvalues for a pair (B, C) has been established in [19] and [20]. In Section 4 we find a related upper bound for QEiCP, and exhibit an example with a large number of them (without attaining however the upper bound).

An enumerative method and a hybrid algorithm, combining the previous method and a semi-smooth approach, have been introduced in [6] and [7]. These algorithms are able to solve the QEiCP when the co-regularity and co-hyperbolicity conditions are assumed to hold. In Section 5 we study the numerical solution of the QEiCP by solving its equivalent EiCP as a Variational Inequality Problem in the $2n$ -dimensional simplex. We propose a projection algorithm discussed in [2] for solving this VIP. The numerical experiments reported in Section 5 indicate that the projection algorithm is able to solve the VIP for many test problems, but may fail to get a solution in some instances. A hybrid enumerative method in the spirit of [7] should be developed in the future for guaranteeing solution of the QEiCP under the sufficient condition presented in Section 2.

2 A sufficient condition for existence of solutions of QEiCP

In this section we will present a new sufficient condition for the existence of solutions of QEiCP and compare it with the one in [21]. The condition is based upon the study of the relation between an arbitrary n -dimensional QEiCP and two specific instances of EiCP with matrices in $\mathbb{R}^{2n \times 2n}$. A relation of this kind was also studied in [5], but the instance of EiCP chosen in this reference is different from the ones in this paper, which are tailored for addressing the existence issue.

For the sake of a simpler notation, we commit a slight notational abuse, and say that a pair $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n$ solves EiCP(B, C) when the triplet (λ, x, w) , with $w = \lambda Bx - Cx$, is a solution of EiCP(B, C) in the sense defined in Section 1. In the same fashion, we say that $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n$ solves QEiCP(A, B, C) when the same occurs with the triplet (λ, x, w) , where $w = \lambda^2 Ax + \lambda Bx + Cx$.

Consider now QEiCP(A, B, C) with $A, B, C \in \mathbb{R}^{n \times n}$ and define $D, G, H \in \mathbb{R}^{2n \times 2n}$ as

$$D = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \quad (9)$$

$$G = \begin{pmatrix} -B & -C \\ I & 0 \end{pmatrix}, \quad (10)$$

$$H = \begin{pmatrix} B & -C \\ I & 0 \end{pmatrix}. \quad (11)$$

Next we establish a relation between the solutions of QEiCP(A, B, C) and those of EiCP(D, G) and EiCP(D, H). We emphasize that the following result holds without making any additional hypotheses on A, B, C .

Proposition 1. *a) Assume that (λ, x) solves QEiCP(A, B, C) and consider D, G, H as in (9)–(11).*

- i) If $\lambda = 0$ then $(\lambda, z) = (0, z)$ solves both $\text{EiCP}(D, G)$ and $\text{EiCP}(D, H)$, where $z \in \mathbb{R}^{2n}$ is defined as $z = (0, x)$.
- ii) If $\lambda > 0$ then (λ, z) solves $\text{EiCP}(D, G)$, where $z \in \mathbb{R}^{2n}$ is defined as $z = (1 + \lambda)^{-1}(\lambda x, x)$.
- iii) If $\lambda < 0$ then the pair $(-\lambda, z)$ solves $\text{EiCP}(D, H)$, where $z \in \mathbb{R}^{2n}$ is defined as $z = (1 - \lambda)^{-1}(-\lambda x, x)$.

b) Consider D, G, H as in (9)–(11).

- i) If (λ, z) solves $\text{EiCP}(D, G)$ with $z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \neq 0$, then $\lambda > 0$ and $(\lambda, (1 + \lambda)x)$ solves $\text{QEiCP}(A, B, C)$
- ii) If (λ, z) solves $\text{EiCP}(D, H)$ with $z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \neq 0$, then $\lambda > 0$ and $(-\lambda, (1 + \lambda)x)$ solves $\text{QEiCP}(A, B, C)$.

Proof. a) For item (i), note that checking whether $(0, x)$ solves $\text{QEiCP}(A, B, C)$ reduces to verifying that $Cx \geq 0$, $x \geq 0$, $x^t Cx = 0$. Hence $(0, (0, x))$ solves both $\text{EiCP}(D, G)$ and $\text{EiCP}(D, H)$. We deal now with item (ii). Note that checking that a pair (λ, z) with $z = (u, v) \in \mathbb{R}^n \times \mathbb{R}^n$ solves $\text{EiCP}(D, G)$ is equivalent to verifying:

$$\lambda Au + Bu + Cv \geq 0, \quad (12)$$

$$\lambda v - u \geq 0, \quad (13)$$

$$u \geq 0, \quad v \geq 0, \quad (14)$$

$$u^t(\lambda Au + Bu + Cv) + v^t(\lambda u - v) = 0, \quad (15)$$

$$e^t u + e^t v = 1. \quad (16)$$

On the other hand, since (λ, x) solves $\text{QEiCP}(A, B, C)$, we know that

$$\lambda^2 Ax + \lambda Bx + Cx \geq 0, \quad (17)$$

$$x \geq 0, \quad (18)$$

$$x^t(\lambda^2 Ax + \lambda Bx + Cx) = 0, \quad (19)$$

$$e^t x = 1. \quad (20)$$

If we take $u = \frac{\lambda}{1+\lambda}x$, $v = \frac{1}{1+\lambda}x$, then (13) follows immediately, and indeed with equality. Furthermore, (12) follows from (17), and (14) follows from (18) and positivity of λ . Also, the first term of the left hand side of (15) vanishes as a consequence of (19) and the second one because $\lambda v = u$. Regarding (16), note that $e^t u + e^t v = (1 + \lambda)^{-1}(\lambda e^t x + e^t x) = e^t x = 1$, using (20) in the third equality. For item (iii), note that if (λ, x) solves $\text{QEiCP}(A, B, C)$ then $(-\lambda, x)$ solves $\text{QEiCP}(A, -B, C)$. In such a case, since $-\lambda$ is positive, we can apply item (ii) to $\text{QEiCP}(A, -B, C)$, replacing λ by $-\lambda$ and B by $-B$. This gives the result, taking into account the definitions of z and H .

b) Consider first item (i). We know that (12)–(16) hold with $(u, v) = (y, x)$, and we need to check that

$$(1 + \lambda)(\lambda^2 Ax + \lambda Bx + Cx) \geq 0, \quad (21)$$

$$(1 + \lambda)x \geq 0, \quad (22)$$

$$(1 + \lambda)^2 [x^t(\lambda^2 Ax + \lambda Bx + Cx)] = 0, \quad (23)$$

$$(1 + \lambda)e^t x = 1. \quad (24)$$

If $\lambda \geq 0$ then (22) follows immediately from (14). It is rather elementary to verify that if it holds that

$$y = \lambda x, \quad (25)$$

then (21) follows from (12), (23) follows from (19), and (24) follows from (20). Therefore $(\lambda, (1 + \lambda)x)$ solves $\text{QEiCP}(A, B, C)$, provided $\lambda \geq 0$. We proceed to prove (25) component-wise, and at the same time we establish that $\lambda \geq 0$. Note that (13) and (14) imply that $x \geq 0, y \geq 0$ and

$$\lambda x - y \geq 0. \quad (26)$$

Taking into account (12), we conclude that the four factors in both terms of the left hand side of (15) are nonnegative, so that both terms vanish. Looking now at the second one component-wise, we have $x_i(\lambda x_i - y_i) = 0$. If $x_i > 0$ then we get $\lambda x_i = y_i$ as required. If $x_i = 0$, we get from (26) that $-y_i \geq 0$, and hence $y_i = 0$ by (14), so that $\lambda x_i = y_i$ holds trivially. We have shown that (25) holds, and hence $(\lambda, (1 + \lambda)x)$ solves $\text{QEiCP}(A, B, C)$. Finally, positivity of λ follows also from (25): since $(x, y) \geq 0$ by (14) and $(x, y) \neq 0$ by (16), $\lambda \leq 0$ entails a contradiction with (25).

For item (ii), we apply the same argument as in item (i) to $\text{EiCP}(D, H)$. Since G and H differ just by the sign of B , we conclude that $(\lambda, (1 + \lambda)x)$ solves $\text{QEiCP}(A, -B, C)$. It now follows from the definition of $\text{QEiCP}(A, B, C)$ that $(-\lambda, (1 + \lambda)x)$ solves it. □

We comment that our sufficient condition requires only item (b) of Proposition 1; however, item (a) has some interesting consequences, see Remarks 1 and 2 below.

Now we rephrase the result of Proposition 1 just in terms of complementary eigenvalues.

Corollary 1. *Consider $\text{QEiCP}(A, B, C)$ with $A, B, C \in \mathbb{R}^{n \times n}$ and the matrices $D, G, H \in \mathbb{R}^{2n \times 2n}$ as defined in (9)–(11). Then,*

- i) each quadratic complementary eigenvalue for (A, B, C) is either a complementary eigenvalue for (D, G) or a complementary eigenvalue for (D, H) ,*
- ii) all nonzero complementary eigenvalues for (D, G) are positive, and are quadratic complementary eigenvalues for (A, B, C) ,*

iii) all nonzero complementary eigenvalues for (D, H) are positive, and their additive inverses are quadratic complementary eigenvalues for (A, B, C) .

Proof. Elementary from Proposition 1. □

Corollary 1 signals a clear path for obtaining a sufficient condition for existence of solutions of $\text{QEiCP}(A, B, C)$: we must first find a sufficient condition for solvability of $\text{EiCP}(D, G)$ or $\text{EiCP}(D, H)$ (which in principle depends only on the matrix in the leading term in (1), namely D in this case, and henceforth just on A , in terms of the data of the QEiCP), and then impose conditions ensuring that either 0 is a quadratic complementary eigenvalue for (A, B, C) , or that 0 is not a complementary eigenvalue of $(D, G), (D, H)$ (which, as mentioned in the proof of Proposition 1(a), depends only upon C).

We will present next some already known conditions which fit the recipe above, for which we need to recall the definitions of three classes of matrices (see e.g. [3]).

Definition 1. i) A matrix $M \in \mathbb{R}^{n \times n}$ is said to be strictly copositive (SC) if $x^t M x > 0$ for all $0 \neq x \in \mathbb{R}^n, x \geq 0$.

ii) The class $R_0 \subset \mathbb{R}^{n \times n}$ consists of those matrices $M \in \mathbb{R}^{n \times n}$ such that there exists no $x \in \mathbb{R}^n \setminus \{0\}$ satisfying $x \geq 0, Mx \geq 0, x^t M x = 0$.

iii) The class $S_0 \subset \mathbb{R}^{n \times n}$ consists of those matrices $M \in \mathbb{R}^{n \times n}$ such that there exists $0 \neq x \in \mathbb{R}^n$ satisfying $x \geq 0, Mx \geq 0$.

It is obvious that 0 is a quadratic complementary eigenvalue for (A, B, C) if and only if M is not R_0 . In view of this observation, we concentrate on the issue of detecting whether (A, B, C) has a nonzero complementary eigenvalue.

Note that verifying whether a matrix C is not R_0 reduces to show that the following nonlinear program:

$$\begin{aligned} & \text{Minimize} && \frac{1}{2} x^t (C + C^t) x \\ & \text{subject to} && Cx \geq 0, e^t x = 1, x \geq 0 \end{aligned}$$

has a global minimum with zero optimal value. So, it is in general difficult to verify whether a matrix is R_0 or not. Furthermore, note that if C is not a S_0 matrix then C is a R_0 matrix. It is also important to add that verifying whether a matrix M is S_0 reduces to see whether the following linear program:

$$\begin{aligned} & \text{Minimize} && c^t x \\ & \text{subject to} && Mx \geq 0, e^t x = 1, x \geq 0 \end{aligned}$$

is feasible for a fixed vector $c \geq 0$ ($c = e$ for instance).

Proposition 2. i) If $M \in \mathbb{R}^{n \times n}$ is strictly copositive then $\text{EiCP}(M, C)$ has solutions for any $C \in \mathbb{R}^{n \times n}$.

ii) If $C \notin S_0$ then 0 is not a complementary eigenvalue for either (D, G) or (D, H) with D, G, H as in (9)–(11).

Proof. Item (i) has been proved in [13] (see also Section 3). For item (ii), assume that 0 is a complementary eigenvalue for (D, G) , with associated complementary eigenvector $0 \neq z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$. It follows immediately that $By + Cx \geq 0$, $-y \geq 0, x \geq 0, y \geq 0$, implying $y = 0$, and hence $Cx \geq 0$ and $x \neq 0$ because $z \neq 0$, so that we get a contradiction with the assumption that $C \notin S_0$. The same argument can be used for the case of (D, H) . \square

Now, all the pieces are in place for stating and proving our existence result for QEiCP.

Theorem 1. *If A is a SC matrix and C is not a S_0 matrix, then (A, B, C) admits at least one positive quadratic complementary eigenvalue and one negative quadratic complementary eigenvalue.*

Proof. Strict copositivity of A implies strict copositivity of D , so that both $\text{EiCP}(D, G)$ and $\text{EiCP}(D, H)$ have complementary eigenvalues by Proposition 2(i), which are nonzero by Proposition 2(ii), and hence positive by items (ii) and (iii) of Corollary 1. Hence there exist at least one positive and one negative quadratic complementary eigenvalue for (A, B, C) . \square

In the remainder of this section, we discuss the existence result given in Theorem 1. We start with a corollary, stating that the roles of A and C in item (ii) of Theorem 1 can be reversed.

Corollary 2. *Consider $\text{QEiCP}(A, B, C)$ and assume that $A \notin S_0$ and C is strictly copositive. Then there exist at least one positive and one negative quadratic complementary eigenvalue for (A, B, C) .*

Proof. Apply Theorem 1(ii) to $\text{QEiCP}(C, B, A)$ and conclude that it has a solution (λ, x) with $\lambda > 0$, so that

$$w = \lambda^2 Cx + \lambda Bx + Ax \geq 0, x \geq 0, w^t x = 0. \quad (27)$$

Let $\mu = \lambda^{-1}$. Divide the first inequality in (27) by λ^2 , and get from (27) $\bar{w} = \mu^2 Ax + \mu Bx + Cx \geq 0, x \geq 0, \bar{w}^t x = 0$, so that (μ, x) solves $\text{QEiCP}(A, B, C)$ and $\mu > 0$. Proceeding in the same fashion with $\text{QEiCP}(C, -B, A)$, get a solution $(\bar{\lambda}, \bar{x})$ of this problem with $\bar{\lambda} > 0$, take $\bar{\mu} = \bar{\lambda}^{-1}$ and conclude that $(\bar{\mu}, \bar{x})$ solves $\text{QEiCP}(A, -B, C)$, so that $-\bar{\mu}$ is a negative quadratic complementary eigenvalue for (A, B, C) . \square

We continue with four remarks related to the result in Theorem 1.

Remark 1. When we move from $\text{QEiCP}(A, B, C)$ to $\text{EiCP}(D, G)$, we can settle the issue of existence of solutions for the former excepting in one “undetermined” case: when we only know that 0 is a complementary eigenvalue for (D, G) . If $\text{EiCP}(D, G)$ has no solutions then the same happens to $\text{QEiCP}(A, B, C)$ by Corollary 1(i), if $\text{EiCP}(D, G)$ has a solution (λ, x) with $\lambda \neq 0$ then λ is a quadratic complementary eigenvalue for (A, B, C) by Corollary 1(ii), but the fact that 0 is a complementary eigenvalue for (D, G) entails no conclusion at all about the existence of solutions of $\text{QEiCP}(A, B, C)$. The same considerations hold for $\text{EiCP}(D, H)$.

Remark 2. Another consequence of Corollary 1 is the following: if a method for finding all complementary eigenvalues for an arbitrary instance EiCP is available, applying it to $\text{EiCP}(D, G)$ and $\text{EiCP}(D, H)$ will provide all quadratic complementary eigenvalues of $\text{QEiCP}(A, B, C)$; in fact all complementary eigenvalues of these two EiCP's will result in quadratic complementary eigenvalues for $\text{QEiCP}(A, B, C)$ (with the possible exception of 0, which can be checked separately) by virtue of Corollary 1(ii)–(iii), and no quadratic complementary eigenvalue will be missed, as a consequence of Corollary 1(i).

Remark 3. We mention that strict copositivity of A by itself is not sufficient for existence of solutions of $\text{QEiCP}(A, B, C)$. Considering $\text{QEiCP}(I, 0, I)$, it is easy to show that it lacks solutions, while I is strictly copositive. In this case $C \in S_0$ and 0 is a complementary eigenvalue for (D, G) , but $C \in R_0$ and hence 0 is not a quadratic complementary eigenvalue of $\text{QEiCP}(I, 0, I)$.

Remark 4. Showing that a matrix C either belongs or does not belong to S_0 reduces to a linear program. Furthermore, any one of the following two conditions is obviously sufficient for ensuring that $C \notin S_0$:

- i) $-C$ is strictly copositive (or even positive definite),
- ii) C has a fully negative row, i.e. there exists $i \in \{1, \dots, n\}$ such that $C_{ij} < 0$ for all $j \in \{1, \dots, n\}$.

Finally, we close the section with the comparison between our sufficient condition for existence of solutions of $\text{QEiCP}(A, B, C)$ and an already known sufficient condition, introduced by A. Seeger in [21] and stated in the next proposition.

Proposition 3. *If (A, B, C) satisfy*

$$x^t Ax \neq 0, \tag{28}$$

$$(x^t Bx)^2 \geq (x^t Ax)(x^t Cx), \tag{29}$$

for all $0 \neq x \in \mathbb{R}^n, x \geq 0$, then $\text{QEiCP}(A, B, C)$ has solutions.

Proof. See [21]. □

In [21], matrices A satisfying (28) are called *co-regular* and triplets (A, B, C) satisfying (29) are said to be *co-hyperbolic*.

For the comparison between the assumptions of Theorem 1 and Proposition 3, we say that a triplet (A, B, C) satisfies (P) when either $C \notin S_0$ and A is strictly copositive, or $C \notin R_0$, and that it satisfies (P') when A is co-regular and (A, B, C) is co-hyperbolic.

We mention that if A is strictly copositive and C satisfies condition (i) in Remark 4, then (P') holds, because in such a case one has $x^t Ax \geq 0$, $x^t Cx \leq 0$ for all $x \in \mathbb{R}_+^n$, so that the right hand side in (29) is nonpositive, making this inequality valid.

On the other hand, it is easy to exhibit instances in which (P) holds but (P') does not. For instance, take A positive definite, $B = 0$ and C satisfying condition (ii) in Remark 4 and having a positive diagonal element (i.e., there exist $i, k \in \{1, \dots, n\}$ such that $C_{ij} < 0$ for all $j \in \{1, \dots, n\}$ and $C_{kk} > 0$). Clearly (P) holds but, taking x equal to the k -th vector in the canonical basis of \mathbb{R}^n , i.e. e^k with $e_j^k = \delta_{jk}$ (Kronecker's delta), one has $x^t B x = 0$, $(x^t A x)(x^t C x) = A_{kk} C_{kk} > 0$. Hence (P') fails.

There are also many instances of QEiCP for which (P') holds but not (P). Take for instance $A = C = I$, $B = 2I$. Validity of (P') is immediate, but (P) fails because $I \in R_0 \cup S_0$, as can be easily verified. Hence, (P) and (P') are independent of each other.

We comment now on some features of (P) and (P').

- i) (P) depends only upon the matrices A and C , while (P') also involves the matrix B .
- ii) The copositivity of A in (P) and the co-regularity of A in (P') are in a certain sense comparable in terms of the difficulty of checking their validity. In fact, co-regularity of A is equivalent to copositivity of either A or $-A$, because it implies that the sign of $\phi(x) = x^t A x$ cannot change within the nonnegative orthant. Both hold when A is positive definite, a standard condition for the matrix in the leading term for EiCP (note that (P') also holds when A is negative definite).
- iii) On the other hand, there is a remarkable difference between (P) and (P') in terms of the remaining conditions, i.e., besides copositivity in (P) and co-regularity in (P'). The co-hyperbolicity condition given by (29) is definitely quite hard to check, excepting in very special cases (e.g. when A and $-C$ are both copositive, as mentioned above). At the same time, determining whether a given matrix belongs to the class S_0 reduces to solving a linear programming problem, a task much easier than determining copositivity, for instance.

Other differences between (P) and (P'), related to the variational inequality problems induced by each of them, are discussed in the next section.

3 The reformulation of QEiCP as nonlinear complementarity or variational inequality problems

The fact that EiCP can be reformulated as a nonlinear complementarity or a variational inequality problem was already mentioned in Section 1, and was in fact recognized very early in the history of the subject, see e.g. [13]. We show in this section that the same reformulations work for QEiCP (see e.g. [5]). We start by giving a general overview of the issue, for the sake of self-containment, and then we discuss it from the perspective of properties (P) and (P').

We recall now the definition of the nonlinear complementarity problem. Let \mathbb{R}_+^m be the non-negative orthant of \mathbb{R}^m . Given $F : \mathbb{R}_+^m \rightarrow \mathbb{R}^m$, the *nonlinear complementarity problem* $NCP(F)$

consists of finding $z \in \mathbb{R}^m$ such that

$$z \geq 0, \quad F(z) \geq 0, \quad F(z)^t z = 0. \quad (30)$$

We rewrite EiCP and QEiCP as instances of NCP. Consider first EiCP(B, C) and assume that B is strictly copositive. Condition (3) can be rewritten as $\lambda(x^t Bx) - (x^t Cx) = 0$, so that if (λ, x) solves EiCP(B, C) then $\lambda = (x^t Cx)/(x^t Bx)$. If we consider now $\bar{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as introduced in Section 1, namely

$$\bar{F}(x) = \frac{x^t Cx}{x^t Bx} Bx - Cx, \quad (31)$$

then it is immediate that (1)–(3) are precisely the conditions defining NCP(\bar{F}), namely (30). Since (4) has been introduced just for ensuring that complementary eigenvectors are nonzero, we can encapsulate the relation between EiCP(B, C) and NCP(\bar{F}) in the following proposition.

Proposition 4. *If (x^*, λ^*) solves EiCP(B, C) then x^* solves NCP(\bar{F}) with \bar{F} as in (31). If \bar{x} is a nonzero solution of NCP(\bar{F}) then (λ^*, x^*) solves EiCP(B, C) with $x^* = \|\bar{x}\|_1^{-1} \bar{x}$, $\lambda^* = (\bar{x}^t C\bar{x})/(\bar{x}^t B\bar{x})$.*

Proof. Elementary; see the paragraph just before the statement of the proposition. \square

We continue with QEiCP(A, B, C), observing that if (λ, x) solves QEiCP(A, B, C) then (7) can be rewritten as $\lambda^2(x^t Ax) + \lambda(x^t Bx) + (x^t Cx) = 0$ so that λ can be obtained from x by solving this quadratic equation. Assume now that A is co-regular and (A, B, C) is co-hyperbolic, and define $\lambda_1, \lambda_2 : \mathbb{R}_+^n \rightarrow \mathbb{R}$ and $F_1, F_2 : \mathbb{R}_+^n \rightarrow \mathbb{R}^n$ as

$$\lambda_1(x) = \frac{-x^t Bx + \sqrt{(x^t Bx)^2 - 4(x^t Ax)(x^t Cx)}}{2x^t Ax}, \quad (32)$$

$$\lambda_2(x) = \frac{-x^t Bx - \sqrt{(x^t Bx)^2 - 4(x^t Ax)(x^t Cx)}}{2x^t Ax}, \quad (33)$$

$$F_i(x) = \lambda_i(x)^2 Ax + \lambda_i(x) Bx + Cx, \quad (i = 1, 2). \quad (34)$$

As in the case of EiCP, we get the following connection between QEiCP(A, B, C) and NCP(F_1), NCP(F_2):

Proposition 5. *If (x^*, λ^*) solves QEiCP(A, B, C) then x^* solves either NCP(F_1) or NCP(F_2), with F_i as in (34) ($i = 1, 2$). If \bar{x} is a nonzero solution of NCP(F_i) ($i = 1, 2$) then (λ^*, x^*) solves QEiCP(A, B, C) with $x^* = \|\bar{x}\|_1^{-1} \bar{x}$, $\lambda^* = \lambda_i(\bar{x})$ ($i = 1, 2$), as defined in (32), (33).*

Proof. Elementary, and similar to the proof of Proposition 4. \square

An undesirable feature of this reduction of EiCP or QEiCP to NCP is that only nonzero solutions of NCP give rise to solutions to EiCP or QEiCP, and this request does not appear in the definition of NCP. It would be desirable to add this request to NCP as an additional constraint, as in (4), (8). The attempt to add additional constraints to a nonlinear complementarity problem leads naturally to a variational inequality problem, where the feasible set can be any closed and convex subset of \mathbb{R}^m , rather than the nonnegative orthant, as is the case for NCP.

We recall that given a closed and convex $K \subset \mathbb{R}^m$ and $F : K \rightarrow \mathbb{R}^m$, the *variational inequality problem* $\text{VIP}(F, K)$ consists of finding $\bar{z} \in K$ such that

$$F(\bar{z})^t(z - \bar{z}) \geq 0 \quad \forall z \in K. \quad (35)$$

For our purposes, we are interested in a particular subset of \mathbb{R}^m as the feasible set for VIP, namely the set Ω introduced in Section 1, defined as $\Omega = \{z \in \mathbb{R}^m : z \geq 0, e^t z = 1\}$, with $e = (1, 1, \dots, 1)^t \in \mathbb{R}^m$. Since Ω is the standard simplex and F is positively homogeneous then [4] $\text{VIP}(F, \Omega)$ becomes

$$z \geq 0, F(z) \geq 0, z^t F(z) = 0, e^t z = 1.$$

We remark that it follows easily from (31)–(34) that the operators \bar{F} , F_1 and F_2 are positively homogeneous and satisfy $0 = \bar{F}(z)^t z, 0 = F_1(z)^t z = F_2(z)^t z$ for all $z \in \Omega$, which means that both for EiCP and for QEiCP the related NCP and VIP are basically equivalent.

As mentioned in Section 1, compactness and convexity of K and continuity of F on K guarantee existence of solutions of $\text{VIP}(F, K)$ (see [4]). This classical result, together with the comments in the previous paragraph and Propositions 4–5, easily provide proofs for Propositions 2(i) and 3.

We have finished with the announced overview, and now we focus on two alternative approaches for solving $\text{QEiCP}(A, B, C)$ through variational inequalities, namely solving $\text{VIP}(F_1, \Omega)$ and $\text{VIP}(F_2, \Omega)$, with $m = n$ and F_1, F_2 as in (34), or solving $\text{VIP}(\bar{F}, \Omega)$ with $m = 2n$ and \bar{F} related to $\text{EiCP}(D, G)$ or $\text{EiCP}(D, H)$, where D, G, H are given by (9)–(11). Taking into account (31), we write next the formula of \bar{F} for this second option in terms of the original data of QEiCP, with $z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$. We have

$$\bar{F}(y, x) = \left(\frac{y^t(I - C)x - y^t B y}{y^t A y + \|x\|_2^2} \right) \begin{bmatrix} A y \\ x \end{bmatrix} + \begin{bmatrix} B y + C x \\ -y \end{bmatrix}. \quad (36)$$

We denote as Approach 1 the one which deals with $\text{VIP}(F_i, \Omega)$ ($i = 1, 2$) and as Approach 2 the one dealing with $\text{VIP}(\bar{F}, \Omega)$, with \bar{F} as in (36).

We proceed to make a conceptual comparison between these two approaches. One obvious drawback of Approach 2 is that it deals with a VIP in dimension $2n$ while both VIP's in Approach 1 work in dimension n .

On the other hand, Approach 2 exhibits at least two attractive features, which might overcome the dimensionality issue. In the first place, the formula of \bar{F} given in (36) is simpler than the formulae of F_1, F_2 given by (32)–(34). In fact, the presence of the square root in (32) and (33) is

certainly undesirable from a numerical point of view, and could make the evaluation of the F_i 's at a given point costlier than an evaluation of \bar{F} , despite the dimensionality issue.

The second advantage of Approach 2 over Approach 1 refers to a robustness property. As we have mentioned, both approaches require similar assumptions on A (co-regularity or strict copositivity) in order to ensure that the denominators in (32), (33) and (36) do not vanish, and when these assumptions are not valid both approaches might fail. The situation is however different when we look at the additional assumptions, in particular at the co-hyperbolicity of (A, B, C) for the case of Approach 1, which, as already mentioned, is pretty hard to check. Assume that we have an instance of QEiCP with strictly copositive A , but such that neither the assumptions on C in Property (P) nor the co-hyperbolicity in Property (P') have been checked. Suppose also that we solve the respective VIP's with some feasible method, i.e. one which approaches a solution z of the problem through a sequence $\{z^k\} \subset \Omega$, and evaluates the operators \bar{F} , F_1 or F_2 at the z^k 's (and possibly at other feasible points too). Regardless of the assumptions on C , when using Approach 2 a solution (λ, z) of EiCP(D, G) or EiCP(D, H) will be found, and if $\lambda \neq 0$ this will provide a solution of QEiCP(A, B, C) as a result of Theorem 1. If $\lambda = 0$, it can be immediately checked whether $(0, z)$ solves QEiCP(A, B, C), and only when this does not happen the procedure fails: it cannot be determined whether (A, B, C) has or not some nonzero quadratic complementarity eigenvalue. The situation is rather worse when following Approach 1. If at some iteration the evaluation of F_1 or F_2 is required at a point x where the co-hyperbolicity condition fails, i.e. such that $(x^T B x)^2 < 4(x^T A x)(x^T C x)$, then the method just breaks when attempting to evaluate the square root in (32) or (33), and nothing is obtained in terms of solutions of QEiCP(A, B, C). In this sense, Approach 2 looks more robust than Approach 1; the former always provides a pair (λ, x) , and a solution of QEiCP whenever $\lambda \neq 0$, while the latter is likely to stop at any iteration when the co-hyperbolicity condition is not known to hold.

4 A nonlinear programming formulation of the QEiCP

Let $A \in SC$ and $C \notin S_0$. If D and G are the matrices given by (9) and (10) respectively, then, by Proposition 1, EiCP (D, G) has at least a solution $(\bar{\lambda}, \bar{z})$ such that $\bar{\lambda} > 0$ and $\bar{z} = (\bar{y}, \bar{x})$ satisfies $\bar{y} = \bar{\lambda}\bar{x}$. Furthermore $(\bar{\lambda}, (1 + \bar{\lambda})\bar{x})$ is a solution of QEiCP. Hence there is a vector $\bar{w} \geq 0$ such that $(\bar{x}, \bar{y}, \bar{w}, \bar{\lambda})$ satisfies the following constraints

$$\lambda \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} - \begin{pmatrix} -B & -C \\ I & 0 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} w \\ 0 \end{pmatrix}$$

$$\begin{aligned} e^t y + e^t x &= 1 \\ y^t w = x^t w &= 0 \\ x, y, w, \lambda &\geq 0 \end{aligned}$$

By introducing the vector \bar{v} defined by $\bar{v} = \bar{\lambda}\bar{y}$, then $(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda})$ is a solution of the following

system

$$Av + By + Cx = w \quad (37)$$

$$e^t y + e^t x = 1 \quad (38)$$

$$e^t v + e^t y = \lambda \quad (39)$$

$$w, x, y, v \geq 0 \quad (40)$$

$$w^t v = w^t y = w^t x = 0 \quad (41)$$

$$y = \lambda x \quad (42)$$

$$v = \lambda y \quad (43)$$

Let

$$K = \{(x, y, v, w, \lambda) : (x, y, v, w, \lambda) \text{ satisfies (37)-(40)}\} \quad (44)$$

and consider the nonlinear program

$$\begin{aligned} \text{NLP :} \quad & \text{Minimize } f(x, y, v, w, \lambda) = \|y - \lambda x\|^2 + \|v - \lambda y\|^2 + (x + y + v)^t w \\ & \text{subject to } (x, y, v, w, \lambda) \in K. \end{aligned} \quad (45)$$

where $\|\cdot\|$ is the Euclidean norm. Then the following result holds:

Proposition 6. *Let $A \in SC$, $C \notin S_0$ and K be the set given by (44). Then the NLP (45) has a global minimum $(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda})$ and $(\bar{\lambda}, (1 + \bar{\lambda})\bar{x})$ is a solution of QEiCP.*

Since computing a global minimum of NLP (45) is a difficult task, it is interesting to investigate when a stationary point of f on K provides a solution of QEiCP. The following result answers to this question.

Proposition 7. *A stationary point $(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda})$ of f on K is a global minimum of NLP (45) with $f(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda}) = 0$ if and only if the Lagrange multipliers associated to the equalities (38) and (39) are equal to zero.*

Proof. Let u, γ_0 and θ_0 be the Lagrange multipliers associated to the equalities (37), (38) and (39) respectively and α, β, γ and θ be the Lagrange multipliers associated to the nonnegative constraints $w \geq 0, x \geq 0, y \geq 0$ and $v \geq 0$ respectively. Hence $(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda})$ satisfies the following KKT conditions:

$$x + y + v = u + \alpha \quad (46)$$

$$-2\lambda(y - \lambda x) + w = -C^t u + \beta + \gamma_0 e \quad (47)$$

$$2[(y - \lambda x) - \lambda(v - \lambda y)] + w = -B^t u + \gamma + \gamma_0 e + \theta_0 e \quad (48)$$

$$2(v - \lambda y) + w = -A^t u + \theta + \theta_0 e \quad (49)$$

$$-2x^t(y - \lambda x) - 2y^t(v - \lambda y) = -\theta_0 \quad (50)$$

$$x, y, v, w \geq 0 \quad (51)$$

$$\beta, \gamma, \theta, \alpha \geq 0 \quad (52)$$

$$\beta^t x = \gamma^t y = \theta^t v = \alpha^t w = 0 \quad (53)$$

Multiplying (46), (47), (48) and (49) by w^t , x^t , y^t and v^t respectively, using (53) and adding the resulting equalities term by term, we get

$$2[(y - \lambda x)^t(y - \lambda x) + (v - \lambda y)^t(v - \lambda y) + w^t(x + y + v)] = \gamma_0 + \theta_0 \lambda$$

i.e.,

$$2f(x, y, v, w, \lambda) = \gamma_0 + \theta_0 \lambda \quad (54)$$

Now, if $\gamma_0 = \theta_0 = 0$, then $f(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda}) = 0$ and $(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda})$ is a global minimum of NLP. Conversely, if $(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda})$ is a global minimum of NLP with $f(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda}) = 0$, then $\gamma_0 + \theta_0 \lambda = 0$. Furthermore $\theta_0 = 0$ by (50) and $\gamma_0 = 0$. \square

Note: A Nonlinear Program similar to NLP (45) can be constructed associated to EiCP (D, H) .

This theorem is not very useful in practice, as it is better to see whether the objective function value at the stationary point is null instead of computing the Lagrange multipliers associated with the equalities (38) and (39). However, it gives an indication that the computation of a simple stationary point may in many cases provide a solution to the QEiCP. This issue is discussed later in this paper.

5 Local algorithms for QEiCP

In Section 3, it is shown that a solution to the QEiCP can be found by solving the $\text{VIP}(\bar{F}, \Omega)$, where \bar{F} is the mapping defined by (36) and

$$\Omega = \{z = (y, x) \in \mathbb{R}^{2n} : e^t y + e^t x = 1, y \geq 0, x \geq 0\}. \quad (55)$$

A popular technique for solving this $\text{VIP}(\bar{F}, \Omega)$ is to use the so-called Regularized Gap-Function f_α defined for a given $\alpha > 0$, by

$$f_\alpha(z) = -\min\{\bar{F}(z)^t(u - z) + \frac{\alpha}{2} \|u - z\|^2 : u \in \Omega\} \quad (56)$$

for each $z = (y, x) \in \Omega$. The following property holds:

Proposition 8. [8, 10] For each $\alpha > 0$, $\bar{z} = (\bar{y}, \bar{x}) \in \Omega$ is a global minimum of

$$\begin{aligned} & \text{Minimize} && f_\alpha(z) \\ & \text{subject to} && z \in \Omega \end{aligned} \tag{57}$$

with value $f_\alpha(\bar{z}) = 0$ if and only if \bar{z} is a solution of $\text{VIP}(\bar{F}, \Omega)$.

This property shows that solving $\text{VIP}(\bar{F}, \Omega)$ reduces to a global optimization problem of a continuously differentiable function on the simplex. Such a point is very difficult to compute in practice. A stationary point of f_α on Ω is much easier to find but there is no guarantee that it provides a solution of the VIP [4]. Furthermore the computation of such a point requires the gradient of f_α , and the computation of this vector is quite involved. To alleviate the computational work we recommend the derivative-free projection algorithm (DFP) discussed in [2] for solving the $\text{VIP}(\bar{F}, \Omega)$. In order to briefly explain such a procedure, let $\bar{z} = (\bar{y}, \bar{x}) \in \Omega$ be a current point. Then a search direction is computed by

$$d = P_\Omega \left(\bar{z} - \frac{1}{\alpha} \bar{F}(\bar{z}) \right) - \bar{z} \tag{58}$$

where $P_\Omega(u)$ is the projection of $u \in \mathbb{R}^{2n}$ in the simplex that is, the unique global minimum and stationary point of

$$\begin{aligned} & \text{Minimize} && \|u - v\|^2 \\ & \text{subject to} && v \in \Omega. \end{aligned} \tag{59}$$

The computation of such a point can be done in polynomial-time by a number of algorithms [2]. Now, if $d = 0$ (i.e., $\|d\| < \epsilon$ for some tolerance) then \bar{z} is a solution of VIP. Otherwise d satisfies $\bar{F}(\bar{z})^t d < 0$ [2] and we look for a stepsize $\delta > 0$ such that implies the reduction of the value of f_α according to the following Armijo type criterion:

$$f_\alpha(\bar{z} + \delta d) \leq f_\alpha(\bar{z}) + \delta \beta \bar{F}(\bar{z})^t d, \tag{60}$$

where $0 < \beta < 1$. As for the usual Armijo criterion, a number of trials of the form

$$\delta = \frac{1}{p^{1.4}}, \tag{61}$$

for $p = 1, 2, \dots$ is done until the condition (60) is satisfied. Unfortunately there is no theoretical guarantee that such a procedure terminates with a stepsize $\delta > 0$ satisfying (60) after a finite number of trials. In this last case the algorithm terminates unsuccessfully. If a stepsize δ is computed then the point \bar{z} is updated by $\tilde{z} = \bar{z} + \delta d$. It is easy to show that this new point \tilde{z} belongs to Ω and a new iteration of the DFP method should be applied with such a point. The steps of the algorithm are presented below:

DFP algorithm

Step 0. Let $\bar{z} = (\bar{y}, \bar{x}) \in \Omega$ and ϵ a positive tolerance.

Step 1. Compute d by (58).

Step 2. If $\|d\| < \epsilon$, terminate with an approximate solution of $\text{VIP}(\bar{F}, \Omega)$.

Step 3. Try to compute a stepsize δ of the form (61) satisfying (60). If such a δ cannot be computed after a finite number of trials terminate the algorithm with a failure.

Step 4. Update $\bar{z} := \bar{z} + \delta d$ and go to Step 1 with the new point \bar{z} .

It is important to note that the verification of the criterion (60) only requires the computation of values of the regularized gap-function f_α . Due to the definition (56) of f_α , the computation of these values reduces to a Strictly Convex Quadratic Separable Program on the simplex, which can be solved in polynomial-time by a number of efficient algorithm [2]. Therefore the DFP method is very simple to implement. The main drawback of this approach is its inability for solving the VIP in general. The performance of the algorithm may be improved by a special choice of the initial point. However, such a choice is also a difficult task.

Another local approach for solving QEiCP consists of finding a stationary point of NLP (45) introduced in Section 4. Such a point can be computed by an active-set method, as that implemented in the well-known code MINOS [15] or any other efficient nonlinear programming algorithm.

In order to verify the efficiency and efficacy of these two local techniques for solving the QEiCP, we have performed some experiments on the solution of the QEiCP test problems discussed in [6] by MINOS and DFP.

For this latter algorithm three different starting points have been used in the experiments:

(INP1) the baricenter of the simplex, i.e.

$$\bar{x}_i = \bar{y}_i = \frac{1}{2n}, \text{ for all } i = 1, \dots, n$$

(INP2) a vector of the canonical basis, i.e.

$$\bar{x}_i = 1, \bar{x}_j = 0, j \neq i$$

$$\bar{y}_j = 0, j = 1, \dots, n$$

for some i (we used $i = n$).

(INP3) the stationary point (\bar{y}, \bar{x}) computed by MINOS.

The numerical performance of these two algorithms is highlighted in Tables 1 and 2 below. In these tables, the following notations are used:

- n : order of the matrices A , B and C .
- f : value of the objective function at the Stationary Point of NLP (45) computed by MINOS.
- λ : value of the variable λ at the Stationary Point of NLP (45) computed by MINOS and the complementary eigenvalue when DFP terminates successfully (i.e., no value of λ is provided when the algorithms were not able to find a solution for QEiCP).
- $\|d\|$: norm of the DFP direction at the termination of the DFP algorithm.
- IT: number of iterations required by the algorithms to terminate.

PROBLEM	n	f	λ	IT
SEEGERADLYQ(3)	3	1.6291E-13	0.2656	4
RAND(0,1,5)	5	4.3760E-17	0.8422	19
RAND(0,1,10)	10	1.5157E-03	0.9057	69
RAND(0,1,20)	20	1.1929E-05	1.0287	231
RAND(0,1,30)	30	9.5175E-04	0.8958	254
RAND(0,1,40)	40	1.6352E-03	0.9045	332
RAND(0,1,50)	50	5.2166E-04	0.9844	538
RAND(0,10,5)	5	6.3680E-03	0.8561	18
RAND(0,10,10)	10	1.8237E-03	0.8937	79
RAND(0,10,20)	20	1.6556E-03	1.0997	99
RAND(0,10,30)	30	1.0369E-02	0.8946	135
RAND(0,10,40)	40	1.8319E-03	0.9983	264
RAND(0,10,50)	50	5.6333E-03	0.7545	365
RAND(0,100,5)	5	8.0186E-17	1.1120	34
RAND(0,100,10)	10	3.6174E-03	0.8139	65
RAND(0,100,20)	20	1.7354E-02	1.0031	75
RAND(0,100,30)	30	7.3798E-03	1.0373	169
RAND(0,100,40)	40	2.8234E-03	0.9097	265
RAND(0,100,50)	50	4.1658E-03	0.9286	287

Table 1: Solution of QEiCP by a Stationary Point of NLP (45).

The following conclusions can be stated from these numerical results:

- (i) MINOS usually finds a stationary point whose objective function value is quite small, i.e., close to a solution of QEiCP. In three cases the stationary point provides a solution to the QEiCP. Note that small values of the objective function at the computed stationary points, such those obtained in this experience, have an important impact on the efficiency of a hybrid enumerative method to be designed in the spirit of [7] for the solution of QEiCP.
- (ii) In general the DFP algorithm does not perform well for each one of the three choices of initial points.

PROBLEM	INP1			INP2			INP3		
	$\ d\ $	IT	λ	$\ d\ $	IT	λ	$\ d\ $	IT	λ
SEEGERADLYQ(3)	9.615E-17	2	0.0	0.0	1	0.0	3.310E-07	1	0.2656
RAND(0,1,5)	3.990E-02	8		9.387E-07	28	0.8253	2.092E-09	1	0.8422
RAND(0,1,10)	9.229E-07	137	0.8107	9.631E-07	130	0.8107	1.884E-02	2	
RAND(0,1,20)	4.834E-07	68	1.1142	0.1231	20		3.399E-03	2	
RAND(0,1,30)	9.830E-07	162	1.0546	0.1100	15		3.072E-02	1	
RAND(0,1,40)	5.031E-02	13		0.2197	4		3.938E-02	2	
RAND(0,1,50)	9.930E-07	315	0.1153	0.1094	13		2.275E-02	1	
RAND(0,10,5)	2.645E-04	16		0.7071	2		7.536E-02	1	
RAND(0,10,10)	0.7046	2		0.6102	3		0.5236	2	
RAND(0,10,20)	0.5419	2		0.9696	3		3.879E-02	1	
RAND(0,10,30)	0.5691	2		0.4518	2		9.243E-02	1	
RAND(0,10,40)	0.4267	2		0.7071	3		3.875E-02	1	
RAND(0,10,50)	0.5833	1		0.7071	3		6.277E-02	1	
RAND(0,100,5)	0.7071	2		0.7071	2		8.932E-09	1	1.1120
RAND(0,100,10)	0.7071	3		0.7071	2		5.536E-02	1	
RAND(0,100,20)	0.7071	2		0.2487	6		0.1180	1	
RAND(0,100,30)	0.6992	1		0.7071	2		2.301E-02	1	
RAND(0,100,40)	0.7655	1		0.7071	3		5.243E-02	1	
RAND(0,100,50)	0.7071	2		0.7071	2		2.628E-02	1	

Table 2: Solution of QEiCP by DFP algorithm.

- (iii) The performance of the DFP algorithm seems to be quite influenced by scaling, as it is much more efficient when all the elements of the matrices A, B and C belong to the interval $[0, 1]$ (four out of six instances have been solved when the barycenter is chosen as the initial point).
- (iv) When DFP algorithm starts with an initial point that is a Stationary Point of NLP (45), then it usually terminates with this point (IT = 1).
- (v) The matrix C of the test problem SEEGERADLYQ(3) is S_0 and is not R_0 . Therefore zero is a complementary eigenvalue of QEiCP. For this instance both the algorithms MINOS and DFP have been able to find a solution of the QEiCP. However, DFP computed the zero complementary eigenvalue while MINOS found a positive eigenvalue.

These results clearly indicate the need of designing an enumerative algorithm similar to the one designed for the EiCP [6, 13] that searches stationary points of NLP in a systematic way until finding one with a null objective function value (or smaller than a prescribed tolerance). As in [7], the efficiency of this enumerative method can be enhanced if it is combined with the DFP method or a semi-smooth algorithm similar to the one discussed in [1] for the EiCP. This will certainly be an interesting topic of our future research.

6 On the number of quadratic complementary eigenvalues

In this section, we discuss the maximum number of quadratic complementary eigenvalues for $\text{QEiCP}(A, B, C)$. We obtain it through the introduction of an additional variable $y \in \mathbb{R}^n$, leading to a $2n$ dimensional problem, similar to the one used in Section 2. A related upper bound for the number of solutions of $\text{EiCP}(B, C)$ has been established in [18].

As mentioned in Section 1, given $B, C \in \mathbb{R}^{n \times n}$, a *generalized eigenvalue* is a complex number λ such that there exists $0 \neq x \in \mathbb{C}^n$ satisfying $\lambda Bx - Cx = 0$. The number of generalized eigenvalues for a pair (B, C) is bounded by n , because the equation above demands singularity of $\lambda B - C$, or equivalently $\det(\lambda B - C) = 0$. Hence, the announced bound follows from the fact that $\det(\lambda B - C)$ is a polynomial of degree at most n as a function of λ , having therefore at most n complex roots.

Based on the same property, it has been proved in Corollary 5.4 of [21] that the number θ_n of quadratic complementary eigenvalues of (A, B, C) , with $A, B, C \in \mathbb{R}^{n \times n}$, is bounded by $n2^n$.

Next we present an n -dimensional instance of QEiCP with $2^{n+1} - 2$ quadratic complementary eigenvalues. This means that $2^{n+1} - 2 \leq \theta_n \leq n2^n$. We comment that an example of an n -dimensional EiCP with $3(2^{n-1} - 1)$ complementary eigenvalues has been exhibited in [23]. This example could be used to generate an example of QEiCP with $3(2^n - 2)$ quadratic complementary eigenvalues, improving over our example. Since the adaptation is not trivial and the example in [23] is itself quite involved, we opted to present our much simpler example, for which the counting procedure is also rather elementary.

Consider \mathbb{R} as a vector space over \mathbb{Q} and take a set $\{r_1, r_2, \dots, r_n\}$ of n positive real numbers which are linearly independent over \mathbb{Q} . Define $r, e \in \mathbb{R}^n$ as $r = (r_1, r_2, \dots, r_n)^t$, $e = (1, 1, \dots, 1)^t$, and consider the matrices A, B, C defined as $A = -I, B = 0, C = er^t$. Note that C has rank 1. For each nonempty subset J of N define the vectors $x^J, w^J \in \mathbb{R}^n$, and the numbers $s^J, \bar{\lambda}^J, \hat{\lambda}^J$ as

$$x_j^J = \begin{cases} 1 & \text{if } j \in J \\ 0 & \text{if } j \notin J, \end{cases} \quad (62)$$

$$s^J = \sum_{j \in J} r_j, \quad \bar{\lambda}^J = \sqrt{s^J}, \quad \hat{\lambda}^J = -\sqrt{s^J}, \quad (63)$$

$$w^J = \bar{\lambda}^2 Ax^J + Cx^J = \hat{\lambda}^2 Ax^J + Cx^J. \quad (64)$$

We claim that all triplets in the set $\{(\bar{\lambda}^J, x^J, w^J)\}_{\emptyset \neq J \subset N} \cup \{(\hat{\lambda}^J, x^J, w^J)\}_{\emptyset \neq J \subset N}$ are solutions of $\text{QEiCP}(A, B, C)$. We proceed to establish the claim.

It follows from (62), (63) that

$$(\bar{\lambda}^J)^2 Ax^J = (\hat{\lambda}^J)^2 Ax^J = -s^J x^J,$$

$$Cx^J = er^t x^J = (r^t x^J)e = s^J e,$$

so that, in view of (64)

$$w^J = s^J(-x^J + e) = s^J(e - x^J). \quad (65)$$

Denoting $J^c = N \setminus J$, it is immediate that $e - x^J = x^{J^c}$, so that $e - x^J \geq 0$ and, taking into account (62), $(x^J)^t(e - x^J) = (x^J)^t x^{J^c} = 0$, so that we conclude, using (65), (64) and the fact that $B = 0$, that

$$\begin{aligned} (\bar{\lambda}^J)^2 Ax^J + \bar{\lambda}^J Bx^J + Cx^J &= (\widehat{\lambda}^J)^2 Ax^J + \widehat{\lambda}^J Bx^J + Cx^J = w^J, \\ x^J \geq 0, \quad w^J \geq 0 \quad (x^J)^t w^J &= 0, \end{aligned}$$

which proves the claim.

No we claim that all the numbers in the set $\{\bar{\lambda}^J\}_{\emptyset \neq J \subset N} \cup \{\widehat{\lambda}^J\}_{\emptyset \neq J \subset N}$ are different. Since we consider only nonempty subsets of N , the $\bar{\lambda}^J$'s are positive, and the $\widehat{\lambda}^J$'s, being the additive inverses of the $\bar{\lambda}^J$'s, are negative, and so it suffices to show that the $\bar{\lambda}^J$'s are all different. We proceed to do so. Assume that $J, K \subset N$ are such that $\bar{\lambda}^J = \bar{\lambda}^K$. Then, by (63),

$$0 = (\bar{\lambda}^J)^2 - (\bar{\lambda}^K)^2 = s^J - s^K = \sum_{j \in J} r_j - \sum_{k \in K} r_k, \quad (66)$$

i.e., we have a linear combination of the r_j 's with rational coefficients (in fact, they are 0, 1 or -1), which vanishes. By the linear independence of the r_j 's, all coefficients in the linear combination must vanish, and then it follows easily from (66) that $J = K$, so that the second claim holds. Since N has $2^n - 1$ nonempty subsets, and we have two quadratic complementary eigenvalues of (A, B, C) for each subset J of N , namely $\bar{\lambda}^J$ and $\widehat{\lambda}^J$, we have proved that (A, B, C) has at least $2^{n+1} - 2$ quadratic complementary eigenvalues.

We observe that the linear independence of the r_i 's over \mathbb{Q} is not essential; it suffice to take the r_i 's so that all their "partial sums" are different. We could have taken, for instance, $r_i = 10^i$.

Observe that the bound for the number of quadratic complementary eigenvalues given by θ_n admits up to $2j$ eigenvalues for each subset J of cardinality j , while our example has only 2, independently of the cardinality of J . In fact, since $\text{rank}(C) = 1$, the same holds for all its principal submatrices, and it is immediate that C^J has one positive eigenvalue, namely s^J , with the associated positive eigenvector e^J (meaning the j -th dimensional version of e), and also the eigenvalue 0, with $j - 1$ eigenvectors, which form a basis of the orthogonal complement of e^J .

It is not difficult to perturb the matrix C so that each submatrix C^J has j different positive eigenvalues, and in such a way that no pair of eigenvalues of different principal submatrices of C coincide, but it does not seem possible to do so while preserving at the same time nonnegativity of all the associated eigenvectors, as well as of all entries of C , and these two nonnegativity conditions seem to be necessary for the satisfaction of the complementarity constraints, in such a way that the eigenvalues of the principal submatrices of C give rise to quadratic complementary eigenvalues of the triplet $(-I, 0, C)$. Possibly, in order to attain the upper bound given by θ_n (if it is at all attainable), an example must be constructed using a triplet (A, B, C) with $(A, B) \neq (-I, 0)$.

References

- [1] Adly, S., Seeger, A. A nonsmooth algorithm for cone constrained eigenvalue problems. *Computational Optimization and Applications* 49 (2011) 299-318.
- [2] Brás, C., Fukushima, M., Júdice, J., Rosa, S. Variational inequality formulation for the asymmetric eigenvalue complementarity problem and its solution by means of a gap function. *Pacific Journal of Optimization* 8 (2012) 197-215.
- [3] Cottle, R.W., Pang, J.S., Stone, R.S. *The Linear Complementarity Problem*. Academic Press, New York (1992).
- [4] Facchinei, F., Pang, J.S. *Finite-dimensional Variational Inequalities and Complementarity Problems*. Springer, Berlin (2003).
- [5] Fernandes, L.M., Júdice, J., Fukushima, M., Iusem, A. On the symmetric quadratic eigenvalue complementarity problem. *Optimization Methods and Software* 29 (2014) 751 - 770.
- [6] Fernandes, L.M., Júdice, J., Sherali, H., Forjaz, M.A. On an enumerative algorithm for solving eigenvalue complementarity problems. *Computational Optimization and Applications* 59 (2014) 113 - 134.
- [7] Fernandes, L.M., Júdice, J., Sherali, H., Fukushima, M. On the computation off all eigenvalues for the eigenvalue complementarity problems. *Journal of Global Optimization* 59 (2014) 307 - 326.
- [8] Fukushima, M. Equivalent differentiable optimization problems and descent methods for variational inequality problems. *Mathematical Programming* 53 (1992) 99-110.
- [9] Golub, G.H., Van Loan, C.F. *Matrix Computations*. John Hopkins University Press, Baltimore (1996).
- [10] Hearn, D. The gap function of a convex program. *Operations Research Letters* 1 (1982) 67-71.
- [11] Júdice, J., Raydan, M., Rosa, S., Santos, S. On the solution of the symmetric complementarity problem by the spectral projected gradient method. *Numerical Algorithms* 44 (2008) 391-407.
- [12] Júdice, J., Sherali, H.D., Ribeiro, I. The eigenvalue complementarity problem. *Computational Optimization and Applications* 37 (2007) 139-156.

- [13] Júdice, J., Sherali, D.H., Ribeiro, I., Rosa, S. On the asymmetric eigenvalue complementarity problem. *Optimization Methods and Software* 24 (2009) 549-586.
- [14] Le Thi, H., Moeini, M., Pham Dinh, T., Júdice, J. A DC programming approach for solving the symmetric eigenvalue complementarity problem. *Computational Optimization and Applications* 5 (2012) 1097-1117.
- [15] Murtagh, B., Saunders, M., Murray, W., Gill, P., Raman, R., Kalvelagen, E., MINOS-NLP. Systems Optimization Laboratory, Stanford University, Palo Alto, CA
- [16] Niu, Y.S., Pham, T. Le Thi, H.A., Júdice, J. Efficient DC programming approaches for the asymmetric eigenvalue complementarity problem. *Optimization Methods and Software* 28 (2013) 812-829
- [17] Pinto da Costa, A., Martins, J., Figueiredo, I., Júdice, J. The directional instability problem in systems with frictional contact. *Computer Methods in Applied Mechanics and Engineering* 193 (2004) 357-384
- [18] Pinto da Costa, A., Seeger, A. Cone constrained eigenvalue problems, theory and algorithms. *Computational Optimization and Applications* 45 (2010) 25-57.
- [19] Queiroz, M., Júdice, J., Humes, C. The symmetric eigenvalue complementarity problem. *Mathematics of Computation* 73 (2003) 1849-1863.
- [20] Seeger, A. Eigenvalue analysis of equilibrium processes defined by linear complementarity conditions. *Linear Algebra and Its Applications* 294 (1999) 1-14.
- [21] Seeger, A. Quadratic eigenvalue problems under conic constraints. *SIAM Journal on Matrix Analysis and Applications* 32 (2011) 700-721.
- [22] Seeger, A., Torkey, M. On eigenvalues induced by a cone constraint. *Linear Algebra and Its Applications* 372 (2003) 181-206.
- [23] Seeger, A., Vicente-Pérez, J. On cardinality of the Pareto spectra. *Electronic Journal of Linear Algebra* 22 (2011) 758-766.