A complementarity eigenproblem in the stability analysis of finite dimensional elastic systems with frictional contact

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Abstract: In this paper a mixed complementarity eigenproblem (MCEIP) is formulated and a method is proposed for its numerical solution. This mathematical problem is motivated by the study of divergence instabilities of static equilibrium states of finite dimensional mechanical systems with unilateral frictional contact. The complementarity eigenproblem is solved by transforming it into a non-monotone mixed complementarity problem (MCP), which is then solved by using the algorithm *PATH*. The proposed method is used to study some small sized examples and some large finite element problems.

1. Introduction

The instability of configurations of static equilibrium of finite dimensional plane linearly elastic systems, in frictional contact with a flat obstacle, and subjected to constant applied forces, was recently discussed by Martins *et al.* [1]. Some of their results were generalised in Martins and Costa [2] for the case of non-linear elastic systems in frictional contact with curved obstacles. In the first of the above papers it is shown that a necessary and sufficient condition for the occurrence of divergence instability along a constant admissible direction, is that an appropriate eigenproblem with the form of an inclusion or a variational inequality is solved, with an appropriate sign of the corresponding real eigenvalue (see Section 2 of the present paper). Still in the paper [1], a necessary condition and a sufficient condition for this type of instability were presented, both of which can be easily checked. However, in many circumstances, those conditions do not yield sufficiently sharp results, in particular the necessary condition frequently gives a poor lower bound estimate for the onset of instability [1]. For this reason we propose in the present paper a procedure for the numerical solution of the governing inclusion or variational eigenproblem. This is achieved by transforming that problem into a complementarity eigenproblem (see Section 3), by means of an appropriate change of the contact related variables (see, for instance, Klarbring [3]). The problem is subsequently transformed into a non-monotone mixed complementarity problem (MCP), in which the unknown eigenvalue is treated as a non-negative variable that is complementary with an additional variable involved in a normalising constraint that prevents the trivial solution (see Section 4). The algorithm *PATH* (Dirkse and Ferris [4]) is successfully applied to solve some small sized examples and some finite element problems (see Section 5).

2. Formulation

We consider a finite dimensional linearly elastic system with plane motion that may establish frictional contact with a fixed flat obstacle, and that is subjected to constant applied forces. The typical situations in mind and the largest examples studied in the present paper involve finite element discretizations of linearly elastic bodies.

For sufficiently smooth time evolutions of the contact candidate particle p of the system, the normal (N) and the tangential (T) components of the vectors (in $\mathbb{R}^2$) of the displacements ($\mathbf{u}_p(t)$), the velocities ($\dot{\mathbf{u}}_p(t)$), and the reactions ($\mathbf{r}_p(t)$) of the particle p satisfy the (Signorini) unilateral contact conditions

$$u_{Np}(t) \leq 0, \quad r_{Np}(t) \leq 0, \quad u_{Np}(t)\, r_{Np}(t) = 0, \tag{1}$$

and the (Coulomb) friction law

$$|r_{Tp}(t)| + \mu\, r_{Np}(t) \leq 0, \quad |\dot{u}_{Tp}(t)|\, r_{Tp}(t) - \mu\, \dot{u}_{Tp}(t)\, r_{Np}(t) = 0, \tag{2}$$

where $\mu \geq 0$ is the coefficient of friction.

We wish to study a particular type of *dynamic* instability of an *equilibrium* state of the system. The smooth portions of its dynamic evolution under constant applied forces $\mathbf{f}^0 \in \mathbb{R}^N$, are governed by the momentum balance equations

$$\mathbf{M}\,\ddot{\mathbf{u}}(t) + \mathbf{K}\,\mathbf{u}(t) = \mathbf{f}^0 + \mathbf{r}(t), \tag{3}$$

where $\mathbf{M}$ is the $N \times N$ symmetric positive definite (SPD) mass matrix, $\mathbf{K}$ is the $N \times N$ SPD stiffness matrix, $t \geq 0$ is the time variable, $(\,\dot{}\,)$ denotes the time derivative, and $\mathbf{u}(t) \in \mathbb{R}^N$ and $\mathbf{r}(t) \in \mathbb{R}^N$ are the vectors of the unknown generalised displacements and reactions at time t, respectively. The system of equations (3) holds together with the unilateral frictional contact conditions (1), (2) at each contact candidate particle p, and together with the absence of reactions along the degrees of freedom that are not subjected to any kinematic constraint: the sub-vector $\mathbf{r}_F(t)$ of $\mathbf{r}(t)$ satisfies

$$\mathbf{r}_F(t) = \mathbf{0}. \tag{4}$$

On the other hand, an equilibrium state of the system under the same applied forces $\mathbf{f}^0$ is characterized by a displacement vector $\mathbf{u}^0 \in \mathbb{R}^N$ and a reaction vector $\mathbf{r}^0 \in \mathbb{R}^N$ that satisfy the equilibrium equations (let $\ddot{\mathbf{u}} \equiv \mathbf{0}$ in (3)):

$$\mathbf{K}\,\mathbf{u}^0 = \mathbf{f}^0 + \mathbf{r}^0, \tag{5}$$

together with the following form of (4) and of the frictional contact conditions at each contact candidate particle p (let $\dot{\mathbf{u}}_p \equiv \mathbf{0}$ in (2)):

$$\mathbf{r}_F^0 = \mathbf{0}, \tag{6}$$

$$u_{Np}^0 \leq 0, \quad r_{Np}^0 \leq 0, \quad u_{Np}^0\, r_{Np}^0 = 0, \quad |r_{Tp}^0| \leq -\mu\, r_{Np}^0. \tag{7}$$

In Martins et al. [1] it is shown that, for t in some right neighbourhood of some instant τ $(t \in [\tau, \tau + \Delta\tau[)$, *there exist dynamic solutions of the form*

$$\mathbf{u}(t) = \mathbf{u}^0 + \alpha(t)\,\mathbf{v}, \qquad \mathbf{r}(t) = \mathbf{r}^0 + \beta(t)\,\mathbf{w}, \tag{8}$$

where $\mathbf{v}$ and $\mathbf{w}$ define constant directions in the sets of right admissible displacement and reaction rates at the equilibrium state $(\mathbf{u}^0, \mathbf{r}^0)$, the function of time α is twice continuously differentiable, α and $\dot{\alpha}$ are non-negative and non-decreasing in $[\tau, \tau + \Delta\tau[$, the function β is continuous, non-negative and non-decreasing in the same interval, and the initial values $\alpha(\tau) \geq 0$, $\dot{\alpha}(\tau) \geq 0$ are arbitrarily small, *if and only if* there exists a number $\lambda \geq 0$, and two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^N$, $\mathbf{v} \neq \mathbf{0}$, such that

$$(\lambda^2 \mathbf{M} + \mathbf{K})\,\mathbf{v} = \mathbf{w},$$
$$\mathbf{w}_f = \mathbf{0},$$
$$\mathbf{v}_d = \mathbf{0},$$

$$
\begin{gathered}
v_{Nsp} = 0, \\
v_{Tsp}\, sign(\overset{0}{r}_{Tsp}) \leq 0, \quad w_{Tsp}\, sign(\overset{0}{r}_{Tsp}) + \mu\, w_{Nsp} \leq 0, \\
[w_{Tsp}\, sign(\overset{0}{r}_{Tp}) + \mu\, w_{Np}]\, [v_{Tsp}\, sign(\overset{0}{r}_{Tsp})] = 0, \\
v_{Nzp} \leq 0,\ w_{Nzp} \leq 0,\ v_{Nzp}\, w_{Nzp} = 0, \\
|w_{Tzp}| + \mu\, w_{Nzp} \leq 0, \quad |v_{Tzp}|\, w_{Tzp} - \mu\, v_{Tzp}\, w_{Nzp} = 0,
\end{gathered} \tag{9}
$$

where the following notations were introduced for the equilibrium state $(\mathbf{u}^0, \mathbf{r}^0)$:

f: degrees of freedom not subjected to any kinematic constraint, including those of the contact candidate particles that currently are not in contact (*free*);

z: particles in contact with *zero* reaction;

d: particles in contact with reaction strictly inside the friction cone and consequent vanishing (right) *displacement* rate;

s: particles in contact with non-vanishing reaction on the friction cone and consequent possible *slip* in the near future.

When the conditions indicated above are satisfied, the equilibrium state corresponding to $\mathbf{u}^0$ and $\mathbf{r}^0$ is dynamically unstable: a divergence instability.

The problem (9) can be equivalently written as an inclusion or a variational inequality eigenproblem (see Martins *et al.* [1]). In order to solve it numerically, we shall write it now as a complementarity eigenproblem.

3. The complementarity eigenproblems

For simplicity of the presentation, we shall restrict ourselves to the transformation of (9) into a complementarity eigenproblem in the particular case in which the set of particles in contact with zero reaction (the particles *z*) is empty. This transformation starts with the elimination of both degrees of freedom of the particles in contact with reaction strictly inside the friction cone (the stick particles *d*), the elimination of the normal degrees of freedom of the particles in contact with non-vanishing reaction on the friction cone (the impending slip particles *s*), and the following change of variables (see also Klarbring [3]):

$$
\mathbf{x} = \begin{Bmatrix} \mathbf{x}_f \\ \mathbf{x}_{Ts} \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{S} \end{bmatrix} \begin{Bmatrix} \mathbf{v}_f \\ \mathbf{v}_{Ts} \end{Bmatrix} = \begin{Bmatrix} \mathbf{v}_f \\ -\mathbf{S}\mathbf{v}_{Ts} \end{Bmatrix} \in \mathbb{R}^{N*}, \tag{10}
$$

$$
\mathbf{y} = \begin{Bmatrix} \mathbf{y}_f \\ \mathbf{y}_{Ts} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ -(\mathbf{S}\mathbf{w}_{Ts} + \mu \mathbf{w}_{Ns}) \end{Bmatrix} \in \mathbb{R}^{N*}, \tag{11}
$$

where $\mathbf{I}$ is an identity (sub-)matrix, $\mathbf{S} = diag(sign(\overset{0}{r}_{Tp}))$ and $N*$ ($\leq N$) is the number of degrees of freedom of the system that may be (right) active at the equilibrium state (the free f plus the impending slip Ts degrees of freedom). In this manner, the following *mixed complementarity eigenproblem in* λ^2 (MCEIP-λ^2) is obtained: *find* $\lambda^2 \geq 0$ *and* $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{N^*} \times \mathbb{R}^{N^*}$, *with* $\mathbf{x} \neq \mathbf{0}$, *such that*

$$(\lambda^2 \mathbf{M}^* + \mathbf{K}^*)\, \mathbf{x} = \mathbf{y}\ , \tag{12}$$

$$\mathbf{y}_f = \mathbf{0}\ , \tag{13}$$

$$\mathbf{0} \leq \mathbf{x}_{Ts} \perp \mathbf{y}_{Ts} \geq \mathbf{0}\ , \tag{14}$$

where $\perp$ denotes orthogonality between the vectors $\mathbf{x}_{Ts}$ and $\mathbf{y}_{Ts}$ in the euclidean inner product. The matrices $\mathbf{M}^*$ and $\mathbf{K}^*$ are linear pencils of matrices in μ,

$$\mathbf{M}^* = \mathbf{M}^*(\mu) = \mathbf{M}_0 + \mu\, \mathbf{M}_1, \qquad \mathbf{K}^* = \mathbf{K}^*(\mu) = \mathbf{K}_0 + \mu\, \mathbf{K}_1\ , \tag{15}$$

with the structure exemplified below for $\mathbf{K}^*$:

$$\mathbf{K}_0 = \begin{bmatrix} \mathbf{K}_{f,f} & -\mathbf{K}_{f,Ts}\mathbf{S} \\ -\mathbf{S}\mathbf{K}_{Ts,f} & \mathbf{S}\mathbf{K}_{Ts,Ts}\mathbf{S} \end{bmatrix}, \quad \mathbf{K}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{K}_{Ns,f} & \mathbf{K}_{Ns,Ts}\mathbf{S} \end{bmatrix}. \tag{16}$$

The matrices $\mathbf{M}_0$ and $\mathbf{K}_0$ in the linear pencils $\mathbf{M}^*$ and $\mathbf{K}^*$ are SPD matrices. In fact $\mathbf{M}_0$ and $\mathbf{K}_0$ are similar to the SPD principal sub-matrices of $\mathbf{M}$ and $\mathbf{K}$ corresponding to the f plus the Ts degrees of freedom, because the diagonal transformation of variables in (10) is orthogonal. By continuity in μ, it follows that, for sufficiently small μ, the matrices $\mathbf{M}^*$ and $\mathbf{K}^*$ are positive definitive (PD). Note also that in the case of a diagonal mass matrix $\mathbf{M}$, the matrix $\mathbf{M}^*$ equals the matrix $\mathbf{M}_0$, so that it is also diagonal and PD.

It is now quite simple to recover the sufficient condition and the necessary condition that were used in [1] to study this problem. *A sufficient condition* for the occurrence of a divergence instability of the form (8) is that there is $\lambda \geq 0$ and $\mathbf{x} \in \mathbb{R}^{N^*}$, with $\mathbf{x} \neq \mathbf{0}$, such that

$$(\lambda^2 \mathbf{M}^* + \mathbf{K}^*)\, \mathbf{x} = \mathbf{0}\ , \tag{17}$$

$$\mathbf{x}_{Ts} \geq \mathbf{0}. \tag{18}$$

Note that this particular case of (12) - (14) results from *a priori* assuming that the solution satisfies $\mathbf{y}_{Ts} = \mathbf{0}$, i.e. the reactions of the contact particles in impending slip (the s particles) remain on the boundary of the friction cone, which means that those particles remain in impending slip or do initiate sliding. Note also that the simplicity of (17), (18) relatively to (12) - (14) is that now (17) is a linear eigenproblem, and the additional inequalities in (18) can be checked *a posteriori*. On the other hand, doing the inner product of

(12) with $\mathbf{x}$, it is immediately seen that *a necessary condition* for the occurrence of a divergence instability of the form (8) is that

$$\textit{the matrix } \mathbf{M}^*(\mu) \textit{ is not } \mathrm{PSD} \textit{ or the matrix } \mathbf{K}^*(\mu) \textit{ is not } \mathrm{PD}. \tag{19}$$

Since, as mentioned above, the matrices $\mathbf{M}^*(\mu)$ and $\mathbf{K}^*(\mu)$ are PD for sufficiently small μ, we can immediately conclude that *no divergence instability of the type* (8) *can occur for sufficiently small coefficients of friction*. Furthermore, in the particular case of a *diagonal mass matrix*, the *necessary condition* (19) reduces to:

$$\textit{the matrix } \mathbf{K}^*(\mu) \textit{ is not } \mathrm{PD}, \tag{20}$$

because, as observed above, $\mathbf{M}^*$ is then diagonal and PD. A related result in a continuum framework can be found in Chateau and Nguyen [5]. Continuing to consider, for simplicity, this particular case of diagonal $\mathbf{M}$ and $\mathbf{M}^*$, the minimum eigenvalue of the symmetric part of $\mathbf{K}^*$ (which is relevant for (20)) is necessarily smaller or equal [1] to the real part of all eigenvalues of $\mathbf{K}^*$ (which are evaluated in (17)); this is one of the main reasons for the necessary condition (20) to be satisfied (much) earlier than the sufficient condition (17).

A related problem that deserves special attention consists of computing the values of the friction coefficient μ and the associated mode shapes that correspond to the transition between stability and instability of a given equilibrium state. This is expressed by the condition $\lambda = 0$ in (12), leading thus to the formulation of a *mixed complementarity eigenproblem in* μ (MCEIP-μ): *find* $\mu \geq 0$ *and* $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{N^*} \times \mathbb{R}^{N^*}$, *with* $\mathbf{x} \neq \mathbf{0}$, *such that*

$$(\mathbf{K}_0 + \mu \mathbf{K}_1)\, \mathbf{x} = \mathbf{y}\ , \tag{21}$$

$$\mathbf{y}_f = \mathbf{0}\ , \tag{22}$$

$$\mathbf{0} \leq \mathbf{x}_{Ts} \perp \mathbf{y}_{Ts} \geq \mathbf{0}\ . \tag{23}$$

Before turning to the numerical solution of the complementarity eigenproblems introduced in this section, we wish to mention an enumerative procedure that yields all the solutions of the mixed complementarity eigenproblems (12) - (14) or (21) - (23). For instance, the problem MCEIP-λ^2 (12) - (14) can be solved by computing the solutions of a set of 2^{n_s} linear eigenproblems, each of them followed by checking some appropriate inequalities; n_s is the number of particles in impending slip at the equilibrium state. Each of those linear eigenproblems is obtained by *a priori* assuming a specific combination of admissible near future evolutions for the *s* particles. Some of those particles are assumed to become *stick* (a subvector

$\mathbf{x}_{Ts}^{\text{stick}}$ of $\mathbf{x}_{Ts}$ is assumed to be null), while the contact reactions of the other ones (the *slip* or impending slip ones) are assumed to remain on the boundary of the friction cone (a subvector $\mathbf{y}_{Ts}^{\text{slip}}$ of $\mathbf{y}_{Ts}$ is assumed to be null). After elimination of the assumed stick variables ($\mathbf{x}_{Ts}^{\text{stick}} = \mathbf{0}$) for each specific combination of near future evolutions, and denoting

$$\bar{\mathbf{x}} = \begin{Bmatrix} \mathbf{x}_f \\ \mathbf{x}_{Ts}^{\text{slip}} \end{Bmatrix}, \tag{24}$$

the problem (12) - (14) can be simplified to a linear eigenproblem of the form

$$(\lambda^2 \bar{\mathbf{M}} + \bar{\mathbf{K}})\, \bar{\mathbf{x}} = \mathbf{0}, \tag{25}$$

followed by the verification of the inequalities

$$\mathbf{y}_{Ts}^{\text{stick}} \geq \mathbf{0},\ \mathbf{x}_{Ts}^{\text{slip}} \geq \mathbf{0}. \tag{26}$$

The small sized example presented in Section 5 is solved by using this enumerative procedure. However, the rapid growth of the number of linear eigenproblems with the number of slip particles makes it impossible to use such method in systems with many contact particles. Finally, it is worth noting that the linear eigenproblem corresponding to the sufficient condition (9) is just *one* of the 2^{n_s} linear eigenproblems considered in this enumerative procedure (25), (26).

4. Numerical solution of the complementarity eigenproblems

In the previous section two complementarity eigenproblems were defined, namely the MCEIP-λ^2 (12)-(14) and the MCEIP-μ (21)-(23). In this section we establish their equivalency to non-linear mixed complementarity problems. This is done by introducing a normalising constraint and an additional non-negative variable (γ) that is complementary to the eigenvalue (λ^2 or μ); this one, in turn, is also considered as an additional non-negative variable.

Proposition 1. The MCEIP-μ defined in (21)-(23) has a solution if and only if there is a solution to the following *mixed complementarity problem* (MCP-μ): *find* $(\mathbf{x}, \mu) \in \mathbb{R}^{N^*+1}$ *and* $(\mathbf{y}, \gamma) \in \mathbb{R}^{N^*+1}$ *such that*

$$(\mathbf{K}_0 + \mu\, \mathbf{K}_1)\, \mathbf{x} = \mathbf{y}, \tag{27}$$

$$\mathbf{e}^{\text{T}}\, \mathbf{x}_{Ts} = c + \gamma, \tag{28}$$

$$\mathbf{y}_f = \mathbf{0}, \tag{29}$$

$$\mathbf{0} \le (\mathbf{x}_{Ts}, \mu) \perp (\mathbf{y}_{Ts}, \gamma) \ge \mathbf{0}, \tag{30}$$

where $\mathbf{e}$ is a vector of dimension n_s will all components equal to 1, and c is an arbitrary positive real number.

Proof. Let $\mathbf{x}$, $\mathbf{y}$ and μ solve the MCEIP-μ. Then $\mathbf{x}_{Ts} \neq \mathbf{0}$, because $\mathbf{x}_{Ts} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$ would imply $\mathbf{x}_f \neq \mathbf{0}$, which is impossible, since the equations in (21) corresponding to the free degrees of freedom f would reduce to

$$\mathbf{K}_{f,f}\, \mathbf{x}_f = \mathbf{0},$$

$\mathbf{K}_{f,f}$ being a SPD matrix. Then $(\mathbf{x}, \mu)$ and $(\mathbf{y}, \gamma)$, with $\gamma = 0$, solve the MCP-μ (27)-(30), with $c = \mathbf{e}^{\mathrm{T}}\, \mathbf{x}_{Ts} > 0$. Note that this positive value of c is arbitrary, because the norm of the vectors $\mathbf{x}$ and $\mathbf{y}$ that solve the eigenproblem (21)-(23) is arbitrary.

Conversely, let $(\mathbf{x}, \mu)$ and $(\mathbf{y}, \gamma)$ solve the MCP-μ (27)-(30). Then $\mathbf{e}^{\mathrm{T}}\, \mathbf{x}_{Ts} = c + \gamma$ for some $c > 0$, so that $\mathbf{e}^{\mathrm{T}}\, \mathbf{x}_{Ts} > 0$, $\mathbf{x}_{Ts} \neq \mathbf{0}$, and, consequently, $\mathbf{x} \neq \mathbf{0}$. Hence $\mathbf{x}$, $\mathbf{y}$ and μ solve the MCEIP-μ (21)-(23).

We can now deduce an equivalent formulation for the MCEIP-λ^2.

Proposition 2. The MCEIP-λ^2 (12)–(14) has a solution if and only if there is a solution to the following *mixed complementarity problem* (MCP-λ^2): *find* $(\mathbf{x}, \lambda^2) \in \mathbb{R}^{N^*+1}$ *and* $(\mathbf{y}, \gamma) \in \mathbb{R}^{N^*+1}$ *such that*

$$(\lambda^2\, \mathbf{M}^* + \mathbf{K}^*)\, \mathbf{x} = \mathbf{y}, \tag{31}$$

$$\mathbf{y}_f = \mathbf{0}, \tag{32}$$

$$\mathbf{e}^{\mathrm{T}}\, \mathbf{x}_{Ts} = c + \gamma, \tag{33}$$

$$\mathbf{0} \le (\mathbf{x}_{Ts}, \lambda^2) \perp (\mathbf{y}_{Ts}, \gamma) \ge \mathbf{0}, \tag{34}$$

where, again, $\mathbf{e}$ is a vector of dimension n_s will all components equal to 1, and c is an arbitrary positive constant.

Proof. Let $\mathbf{x}$, $\mathbf{y}$ and λ^2 be a solution to the MCEIP-λ^2. Then $\mathbf{x}_{Ts} \neq \mathbf{0}$, because $\mathbf{x}_{Ts} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$ would imply $\mathbf{x}_f \neq \mathbf{0}$, which is impossible, since the equations in (12) corresponding to the free degrees of freedom f would reduce to

$$(\lambda^2\, \mathbf{M}_{f,f} + \mathbf{K}_{f,f})\, \mathbf{x}_f = \mathbf{0},$$

with $\mathbf{M}_{f,f}$ and $\mathbf{K}_{f,f}$ both SPD and $\lambda^2 \geq 0$. Then $(\mathbf{x}, \lambda^2)$ and $(\mathbf{y}, \gamma)$, with $\gamma = 0$, solve the MCP-λ^2 (31)-(34), with $c = \mathbf{e}^{\mathrm{T}}\, \mathbf{x}_{Ts} > 0$. Note that the observation and the arguments presented in the final part of the proof of Proposition 1 can also be applied to conclude the present proof.

Remark. Since $\mathbf{K}_0$ is SPD, any solution to the MCEIP-μ has $\mu > 0$ (recall also section 3). Consequently the complementary variable γ vanishes, so that the choice of a positive constant c as data for the MCP-μ specifies the value of $\mathbf{e}^{\mathrm{T}}\, \mathbf{x}_{Ts}$, i.e., the l_1 norm of the vector $\mathbf{x}_{Ts}$. In the case of the MCP-λ^2 the same conclusion can be obtained when μ is not a solution to the MCEIP-μ, because in this situation $\lambda^2 \neq 0$ and, again, the complementary variable γ equals zero.

It may be of interest to search for solutions to the MCEIP-μ with μ below or above a certain prescribed value (μ_0). For that purpose one has to change a non-negative variable in the MCP-μ (27)-(30): in order to search for solutions with $\mu \leq \mu_0$ (or $\mu \geq \mu_0$), one has to consider the new variable $\zeta = \mu_0 - \mu$ (or $\zeta = \mu - \mu_0$), and then to replace μ in (27) by $\mu_0 - \zeta$ (or $\mu_0 + \zeta$), and to replace μ in (30) by ζ. Similar procedures can also be followed for the MCEIP-λ^2.

Propositions 1 and 2 show that the solution of the eigenproblems under study can be obtained by processing two mixed non-linear complementarity problems. There are a number of algorithms for solving these problems. The reader can find in [6] a list of some relevant approaches for non-linear complementarity problems (NCP). More recently, interior-point algorithms [7, 8, 9] and a Newton's method for solving systems of non-differentiable equations [10], based on the so called Fischer function, have also been recommended for this type of problems. The latter algorithms are usually quite efficient, but unfortunately they require the monotonicity of the function, or some similar property, to be useful. The complementarity problems discussed in this section do not share this property. The algorithm *PATH* described in [4] is an algorithm that can process non-monotone mixed complementarity problems and then has been our choice to process the problems discussed in this section. We recall that *PATH* is a robust *GAMS* implementation [11] of a path following technique that was first discussed in Ralph [12] and later improved by Dirkse and Ferris [11]. This algorithm exploits the equivalence of a mixed complementarity problem with a system of nondifferentiable equations $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, where $\mathbf{F}$ is the so-called normal map due to Robinson [13]. The zero of the function is computed by a path generation technique that in each iteration pursues a root of a linear approximation of the normal map at the current iterate. This is done by using a pivotal scheme similar to the well-known Lemke algorithm [14, 15]. A non-monotone line-search technique [16] is also included to guarantee sufficient decrease of the Euclidean norm of the normal map. The algorithm possesses strong global convergence properties [4]. As discussed in the next section, the algorithm has been able to process all the non-linear complementarity problems tried so far that were known to have some solution. A theoretical investigation of this behaviour is certainly a subject for future research.

5. Examples and numerical results

The first example involves two particles of mass m each, supported by linear elastic springs of stiffness k, and in contact with an horizontal obstacle (see Fig. 1). The same example was discussed earlier by Alart and Curnier [17] in the context of non-uniqueness of solution to incremental quasi-static

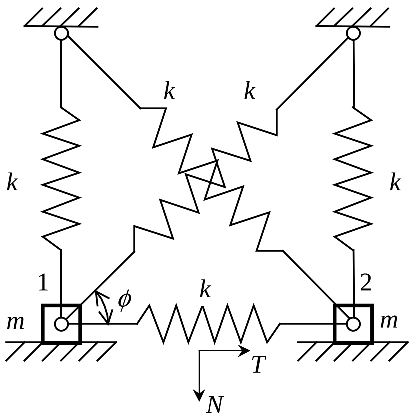

Figure 1. A structure with two contact particles leading to multiple solutions of MCEIP-λ^2.

problems with friction.

The system has four degrees of freedom and the generalised displacements $\mathbf{u} = (u_{T1}, u_{N1}, u_{T2}, u_{N2})$ are used. The external applied forces $\mathbf{f}_1$ and $\mathbf{f}_2$ are such that both particles are in a state of impending slip towards the right or towards the left. The mass and stiffness matrices of the system are, respectively,

$$\mathbf{M} = m\,\mathbf{I}\,, \qquad \mathbf{K} = k\begin{bmatrix} c^2+1 & -cs & -1 & 0 \\ -cs & s^2+1 & 0 & 0 \\ -1 & 0 & c^2+1 & cs \\ 0 & 0 & cs & s^2+1 \end{bmatrix},$$

Table 1. Static equilibrium state, conditions on the data and solutions of MCEIP-λ^2 for the structure with two contact particles of Fig. 1

<table>
<tr><th>S_1</th><th>S_2</th><th>Conditions on the data</th><th colspan="2">Solutions of MCEIP-λ^2</th></tr>
<tr><td rowspan="6">+1</td><td>+1</td><td>$c^2 + 2 - (\mu s)^2 \leq 0$</td><td colspan="2">$\lambda^2 = \frac{k}{m}\left[-(c^2+1) + \sqrt{1+(\mu cs)^2}\right]$
$x_{T1} > 0$ arbitrary, $x_{T2} = \frac{x_{T1}}{\sqrt{1+(\mu cs)^2}+\mu cs}$
$y_{T1} = y_{T2} = 0$ (SLIP - SLIP)</td></tr>
<tr><td rowspan="5">−1</td><td rowspan="2">$c^2 + 1 - \mu cs \leq 0$</td><td rowspan="2">$\lambda^2 = -\frac{k}{m}(c^2 + 1 - \mu cs)$</td><td>$x_{T1} > 0$ arbitrary;
$x_{T2} = 0$;
$y_{T1} = 0$;
$y_{T2} = k\, x_{T1}$
(SLIP - STICK)</td></tr>
<tr><td>$x_{T1} = 0$;
$x_{T2} > 0$ arbitrary;
$y_{T1} = k\, x_{T2}$;
$y_{T2} = 0$
(STICK - SLIP)</td></tr>
<tr><td rowspan="3">$c^2 + 2 - \mu cs \leq 0$</td><td rowspan="2">$\lambda^2 = -\frac{k}{m}(c^2 + 1 - \mu cs)$</td><td>$x_{T1} > 0$ arbitrary;
$x_{T2} = 0$;
$y_{T1} = 0$;
$y_{T2} = k\, x_{T1}$
(SLIP - STICK)</td></tr>
<tr><td>$x_{T1} = 0$;
$x_{T2} > 0$ arbitrary;
$y_{T1} = k\, x_{T2}$;
$y_{T2} = 0$
(STICK - SLIP)</td></tr>
<tr><td>$\lambda^2 = -\frac{k}{m}(c^2 + 2 - \mu cs)$</td><td>$x_{T1} = x_{T2}$
$= x > 0$ arbitrary;
$y_{T1} = y_{T2} = 0$
(SLIP - SLIP)</td></tr>
<tr><td rowspan="2">−1</td><td>+1</td><td>–</td><td colspan="2">No Solution</td></tr>
<tr><td>−1</td><td>$c^2 + 2 - (\mu s)^2 \leq 0$</td><td colspan="2">$\lambda^2 = \frac{k}{m}\left[-(c^2 + 1) + \sqrt{1+(\mu cs)^2}\right]$
$x_{T1} = \frac{x_{T2}}{\sqrt{1+(\mu cs)^2}+\mu cs}$, $x_{T2} > 0$ arbitrary;
$y_{T1} = y_{T2} = 0$ (SLIP - SLIP)</td></tr>
</table>

where **I** denotes the 4 x 4 identity matrix, $s = sin\ \phi$, $c = cos\ \phi$, and ϕ is the angle between the inclined springs and the horizontal.

For the previously described system, the problem MCEIP-λ^2 is: find $\lambda^2 \geq 0$ and $(x_{T1}, x_{T2}, y_{T1}, y_{T2})$, with $(x_{T1}, x_{T2}) \neq (0, 0)$, such that

$$\begin{bmatrix} \lambda^2 m + k\,(c^2+1-\mu cs\, S_1) & -S_1 S_2 k \\ -S_1 S_2 k & \lambda^2 m + k\,(c^2+1+\mu cs\, S_2) \end{bmatrix} \begin{Bmatrix} x_{T1} \\ x_{T2} \end{Bmatrix} = \begin{Bmatrix} y_{T1} \\ y_{T2} \end{Bmatrix}$$

$$0 \leq \begin{Bmatrix} x_{T1} \\ x_{T2} \end{Bmatrix} \perp \begin{Bmatrix} y_{T1} \\ y_{T2} \end{Bmatrix} \geq 0,$$

where $S_p = sign(r^0_{Tp})$, and x_{Tp} and y_{Tp} are defined as in (10) and (11).

Four different combinations of signs for the tangential reactions exist, each one corresponding to a different static equilibrium state. Table 1 contains the solution sets of the above MCEIP-λ^2, for each of those combinations of signs of the static tangential reactions. When both particles are in a state of impending slip to the same side, there is at most one solution; when the left particle is in impending slip towards the right and the right one is in impending slip towards the left, no divergence instability of the type (8) is found; when the left particle is in impending slip towards the left and the right particle is in impending slip towards the right, then, depending on the data, there are at most two or three solutions.

As mentioned earlier, all the solutions for this small sized mechanical system could be calculated by the enumerative procedure presented in the end of section 3. But the algorithm *PATH* was also tested in these small problems. For the sets of numerical data used, the algorithm *PATH* was always able to find the corresponding complete solution set presented in Table 1. For the particular cases that have two solutions with different values of λ^2, the technique mentioned in Section 4 for searching solutions with λ^2 below or above a certain fixed value of λ^2 was successfully applied.

The second example involves a rectangular polyurethane block sliding on an araldite obstacle that was studied experimentally by Villechaise and Zeghloul ([18], [19]). In the numerical simulations we assume that the elastic block slides on a flat rigid obstacle. The block is discretized with a uniform mesh of 800 linear P1 finite elements that has 21 contact candidate nodes (see Fig. 2). The elastic properties are: modulus of elasticity = 5 MPa, Poisson's ratio = 0.48. The geometric parameters are length L = 80 mm, height H = 40 mm and thickness = 9.6 mm. The density of the material is 1.2 kg/dm^3.

The block is submitted to a quasi-static loading consisting first of prescribed displacements on the side CD, which is symmetrically pressed against the obstacle until the resultant of the normal reactions on side AB is -55N. Then the loading proceeds by prescribing an horizontal motion of the side CD towards the left. In this tangential loading phase, the successive equilibrium states have a growing region of nodes in impending slip spreading from right to left.

This same example was studied earlier in [1], also by the finite element method, but using only the necessary (19) and the sufficient (17), (18) conditions for divergence instability recalled in Section 3 of the present paper. For the value of the coefficient of friction ($\mu = 1.1$) identified from the experimental results of Villechaise and Zeghloul ([18], [19]), the numerical results showed that the necessary condition (19) is satisfied very early along the tangential loading process; however, for all the successive equilibrium configurations of the block along that tangential loading, the sufficient condition (17), (18), that involves slip of all nodes in impending slip, could never be satisfied. The objective of the continuation of that study in the present paper is thus to check if, after the necessary condition is satisfied, there exist or not instability modes of a type different from the all-slip modes of the sufficient condition (17), (18).

With this purpose, we search first for solutions to the MCEIP-μ at the equilibrium states obtained with $\mu = 1.1$, along the tangential loading of the block, i.e. we search for the values of the coefficient of friction that would originate a transition from stability to instability in that equilibrium configuration. Nontrivial eigenvectors of MCEIP-μ were obtained when 11 or more contact nodes were in a state of impending slip. It is found that the values of μ that solve the MCEIP-μ decrease with the increase of the number of nodes in impending slip in the successive equilibrium configurations. Moreover, the eigenvectors of the MCEIP-μ associated with higher values of μ correspond to modes having, in average, a larger number of impending slip nodes that get stuck.

For an equilibrium state having the two left nodes stuck**,** the 15 intermediate nodes in impending slip and the 4 nodes on the right free, the algorithm *PATH* converged to a solution of the MCEIP-μ. That solution has a very large value of μ (60.81) and a mode represented in Fig. 2, where an impending slip node (the fourth from the left) becomes stuck. For the same equilibrium configuration, the classical eigenproblem corresponding to the sufficient condition (17), (18) was solved, showing that no positive λ exists that corresponds to an admissible non-trivial solution with all impending slip nodes (*s*) in impending slip or in slip (the sufficient condition (17), (18) could not be satisfied).

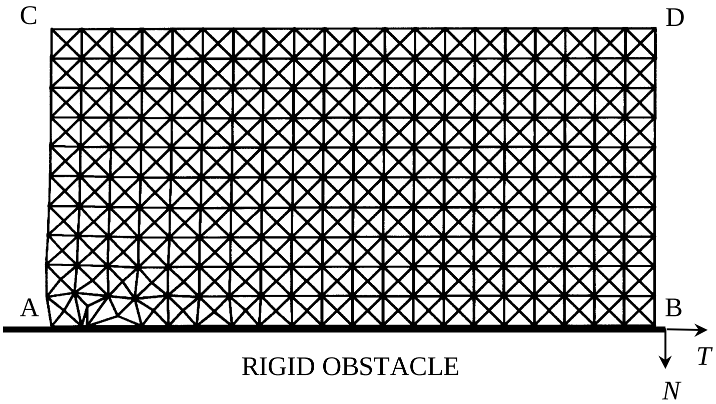

Figure 2. An instability mode in the transition between stability and instability for $\mu = 60.81$ (solution of MCEIP-μ)

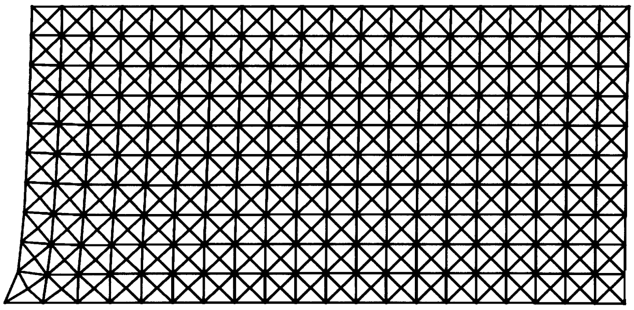

Figure 3. An instability mode in the transition between stability and instability for $\mu = 1.71$ (solution of MCEIP-μ)

For the final equilibrium state of the loading process, for which the 17 nodes on the left are in impending slip and the 4 nodes on the right are free, a nontrivial eigenvector could be found for a much lower coefficient of friction ($\mu = 1.71$). The corresponding divergence eigenmode is represented in Fig. 3. Since all the nodes in impending slip do slide, this mode is given by the sufficient condition (17), (18) (an all-slip mode). For the same equilibrium configuration and choosing a $\mu > 1.71$ the MCEIP-λ^2 has a similar non-trivial eigenvector and a positive eigenvalue $\lambda^2 > 0$.

For other meshes and other aspect ratios H/L, the same trends were observed in the behaviour of the system.

6. Conclusions

In this paper a method to solve a mixed complementarity eigenproblem (MCEIP) has been proposed. The motivation to study this mathematical problem was the divergence instability of static equilibrium states of mechanical systems with unilateral frictional contact. The complementarity eigenproblem has been transformed into a non-monotone mixed complementarity problem (MCP), and the algorithm *PATH* has been applied to solve small sized examples and large finite element problems.

- In all the small sized examples, all the existing solutions could be obtained with the *PATH* algorithm.
- In the large finite element simulations with the block of Zeghloul and Villechaise [18], [19], it has been observed that:
  - whenever solutions were known to exist [the *all-slip* solutions of the sufficient condition (17), (18)] the *PATH* algorithm always converged to one such *all-slip* solution;
  - in some cases where *all-slip* solutions did not exist [the sufficient condition (17), (18) could not be satisfied] other solutions with slip and stick were obtained, but only for very large values of μ;
  - for reasonably small values of μ, no solutions different from the *all-slip* solutions provided by the sufficient condition (17), (18) were found.

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