SOLUTION METHODS FOR STRUCTURAL OPTIMIZATION IN CONTACT ROD PROBLEMS

Isabel N. Figueiredo
Departamento de Matemática, Universidade de Coimbra
Apartado 3008, 3001-454 Coimbra, Portugal

Joaquim J. Júdice
Departamento de Matemática, Universidade de Coimbra
Apartado 3008, 3001-454 Coimbra, Portugal

Silvério S. Rosa
Departamento de Matemática, Universidade da Beira Interior
6201-001 Covilhã, Portugal

Abstract We consider a material and shape optimization problem which involves a composite rod in contact with a rigid foundation. It is shown that the problem can be reduced to an optimization problem in some independent variables and some variables defined implicitly by the solution of a linear complementarity problem (LCP) with a positive definite matrix. A projected-gradient algorithm is proposed that incorporates an efficient LCP solver to compute function values and gradients. An example is included to illustrate the suitability of the proposed methodology.

1. Introduction

We address a structural optimization problem which involves a unidirectional fiber reinforced composite rod in contact with a rigid foundation. The purpose of the model is to find the fiber volume fraction and the size of some geometric parameters of the rod in order to minimize the compliance of the structure. The variables are continuous and vary in an admissible set.

The discretization of this optimization problem by using appropriate finite elements leads to a mathematical programming problem with equilibrium constraints (MPEC), in which the objective function depends on the so-called outer independent variables (material and geometric parameters) and on inner dependent variables, that are the solution of a linear complementarity problem (LCP), representing the contact rod problem. Due to the definition of the compliance function, it is possible to reduce the problem to an optimization problem in the independent variables. Furthermore the constraints on these variables are simple lower and upper bounds and the objective function is continuously differentiable in its constraint set. A projected-gradient algorithm is proposed for the solution of such an optimization problem. In addition it is shown that the values of the objective function and of the gradient in each point used by the algorithm can be obtained by solving special LCPs with positive definite matrices.

The outline of the upcoming sections is as follows. In section 2 the optimization rod problem is introduced. The algorithm and its implementation are discussed in section 3. An example of the application of the algorithm is included in section 4.

2. Notations and Description of the Problem

Let ω be an open, bounded and connected subset of R^2 and $L > 0$ be a constant. We denote by $\overline{\omega} \times [0, L]$ the set occupied by the rod, in its reference configuration, with length L and cross section ω. We assume that the material of the rod is an unidirectional fiber reinforced composite material. We denote by $x = (x_1, x_2, ..., x_s) \in R^s$ the vector whose s components specify the type and the number of material and geometric features of the rod under consideration. Moreover we suppose that the rod is clamped at its extremities and is subjected to the action of applied forces that force a part of the lateral surface of the rod to be in contact with a rigid foundation. In addition we assume that the candidate lateral contact surface is plane and perpendicular to one of the inertia axes of the rod. For a fixed vector x, the continuous one-dimensional equilibrium model describing this contact rod problem is a generalization of the Bernoulli-Navier model and it can be mathematically justified by the asymptotic expansion method, as in Trabucho and Viãno (1996) chap.6, for the homogeneous and isotropic case. Using the finite element method, the discrete formulation, for each x, of the one-dimensional contact rod model under consideration constitutes the following discrete variational inequality

$$\left\{ \begin{array}{l} \text{Find} \quad u \in \overline{U} = \{v \in R^n : \quad v_{\tilde{J}} = 0, \quad v_J \geq \overline{g}_J\}, \quad \text{such that} \\ (v-u)^T (B(x)u - \overline{F}(x)) \geq 0, \quad \forall v \in \overline{U}. \end{array} \right. \tag{1}$$

In (1) n denotes the number of global degrees of freedom of the finite element mesh of the rod axe $[0, L]$. The matrix $B(x)$ is the stiffness matrix and $\overline{F}(x)$ is the vector associated to the applied forces. $B(x)$ depends explicitly on x and $\overline{F}$ may also depend on the components of x. The vector u is the solution of the contact rod model and represents the approximate displacement of the rod. We remark that u depends implicitly on x. The notation $(v-u)^T$ stands for the transpose of vector $(v-u)$. The set $\overline{U}$ is the set of admissible displacements and the sets $\tilde{J}$ and J are subsets of the global degrees of freedom $\{1, 2, ..., n\}$. The vectors $v_{\tilde{J}}$ and v_J are subvectors of v, with components $(v_{\tilde{J}})_{i \in \tilde{J}}$ and $(v_J)_{j \in J}$, respectively. The condition $v_{\tilde{J}} = 0$ corresponds to the clamped rod condition. The vector $\overline{g}_J = (\overline{g}_j)_{j \in J}$ is independent of x and defines the gap between the rod and the rigid foundation at the nodes $j \in J$. The condition $v_J \geq \overline{g}_J$ means that $v_j \geq \overline{g}_j$, for $j \in J$, and states that the rod can touch but not penetrate the rigid foundation at node $j \in J$.

In order to clarify the dependence on x of B and $\overline{F}$ we describe next the structure of the element stiffness matrix and of the element vector force. We denote by h_i the amplitude of the generic finite element $[y_i, y_{i+1}]$ subset of $[0, L]$. Then the corresponding element stiffness matrix B_i is

$$B_i(x) = E \begin{bmatrix} \frac{|w|}{h_i} & 0 & 0 & -\frac{|w|}{h_i} & 0 & 0 \\ 0 & \frac{12I}{h_i^3} & \frac{6I}{h_i^2} & 0 & -\frac{12I}{h_i^3} & \frac{6I}{h_i^2} \\ 0 & \frac{6I}{h_i^2} & \frac{4I}{h_i} & 0 & -\frac{6I}{h_i^2} & \frac{2I}{h_i} \\ -\frac{|w|}{h_i} & 0 & 0 & \frac{|w|}{h_i} & 0 & 0 \\ 0 & -\frac{12I}{h_i^3} & -\frac{6I}{h_i^2} & 0 & \frac{12I}{h_i^3} & -\frac{6I}{h_i^2} \\ 0 & \frac{6I}{h_i^2} & \frac{2I}{h_i} & 0 & -\frac{6I}{h_i^2} & \frac{4I}{h_i} \end{bmatrix} \tag{2}$$

where E, $|w|$ and I depend on x and represent respectively the longitudinal modulus of the material, the area of the cross section and the moment of inertia. Assuming now that q and p are the uniformly distributed forces per unit of length in the direction of the rod axis and in the direction perpendicular to the rod axis, respectively, then the element vector force $\overline{F}_i(x)$ is defined by

$$\overline{F}_i(x)^T = \left[\frac{qh_i}{2} \quad \frac{ph_i}{2} \quad \frac{ph_i^2}{12} \quad \frac{qh_i}{2} \quad \frac{ph_i}{2} \quad -\frac{ph_i^2}{12} \right]. \tag{3}$$

We remark that (1) is an obstacle problem. In particular it can be reformulated as a mixed complementarity problem. To see this, we denote by K and H the subsets of indices defined by $K = \{1, 2, ...n\} \setminus \{\tilde{J} \cup J\}$ and $H = K \cup J$, respectively. By performing the change of variables

$$v \in \overline{U} \iff v - g \in U = \{v \in R^n : \quad v_{\tilde{J}} = 0, \quad v_J \geq 0\} \tag{4}$$

where the vector $g \in R^n$ is defined by

$$g = (g_j)_{j \in R^n} \quad \text{and} \quad g_j = 0, \quad \text{if } j \notin J, \quad g_j = \overline{g}_j, \quad \text{if } j \in J, \tag{5}$$

then problem (1) is equivalent to the following mixed complementarity problem

$$\begin{cases} \text{Find} \quad u \in R^{|H|}, \quad w \in R^{|H|} \quad \text{such that} \\ A(x)u - F(x) = w \\ u_J \geq 0, \quad w_J \geq 0, \quad w_K = 0, \\ u_J^T w_J = 0. \end{cases} \tag{6}$$

The number $|H|$ is the cardinal of H, A is a submatrix of B and F is a subvector of $\overline{F} - Bg$, whose elements have indices in H, that is,

$$A(x) = B_{HH}(x) \qquad \text{and} \qquad F(x) = (\overline{F}(x) - B(x)g)_H. \tag{7}$$

We remark that u is a solution of (6) if and only if $u + g$ is a solution of problem (1).

The structural optimization problem considered in this paper consists of finding an equilibrium point of the contact rod problem that minimizes the compliance function and therefore maximizes the stiffness of the structure. Due to the equivalence shown above, this problem can be written as the following mathematical programming problem with equilibrium constraints (MPEC):

$$MPEC \left[\begin{array}{l} \min \theta(x,u) = \min \frac{1}{2} u^T A(x) u \\ \text{subject to}: \\ x \in X \quad \text{and} \quad \begin{cases} u \in R^{|H|}, \quad w \in R^{|H|} \\ A(x)u - F(x) = w \\ u_J \geq 0, \quad w_J \geq 0, \quad w_K = 0 \\ u_J^T w_J = 0. \end{cases} \end{array} \right. \tag{8}$$

The set X is the set of admissible material and geometric parameters defined by $X = \{x = (x_1, ..., x_s) \in R^s : \ x_i^{\min} \leq x_i \leq x_i^{\max}, \ i = 1, ..., s\}$, where $x_i^{\min}$ and $x_i^{\max}$ are real constants. The objective function $\theta(x, u)$ satisfies

$$\theta(x,u) = \frac{1}{2} u^T A(x) u = \frac{1}{2} u^T (w + F(x)) = \frac{1}{2} u^T F(x) \tag{9}$$

because of the complementarity condition $u^T w = 0$. Therefore, for each x, $\theta(x, u)$ is the compliance of the rod constrained by the zero obstacle and subjected to the action of applied loads represented by the vector $(\overline{F}(x) - B(x)g)_H$. If $g = 0$ then $\theta(x, u)$ is exactly the compliance of the rod with applied forces $\overline{F}(x)$ and constant zero obstacle. We refer to Petersson (1995) for a justification of other definitions of stiffness measure in structural optimization.

3. A Projected-Gradient Algorithm for the MPEC

Consider the inner complementarity problem of the MPEC. Since for each $x \in X$ the matrix $A(x)$ is symmetric positive definite, then this complementarity problem has a unique solution. Hence it is possible to write the MPEC as the following optimization problem in the variable x

$$\left[\begin{array}{l} \min f(x) = \min \theta(x, u(x)) \\ \text{subject to} \quad x \in X \end{array}\right. \tag{10}$$

where u depends implicitly and uniquely on x through the complementarity problem. In general, the non-smoothness of $u(x)$ with respect to the variable x may originate the non-smoothness of the objective function f. In fact, u is a Lipschitz function on the feasible set X, the directional derivative $u'(x, \tilde{x})$ of u at x in the direction $\tilde{x}$ exists, but the gradient $\nabla_x u(x)$ of u at x does not exist if the coincidence set $\{j \in J : \ w_j(x) = 0, \ u_j(x) = 0\}$ is not empty (see, Harker and Pang (1990) or Haslinger and Neittaanmäki (1997) for a justification of these statments). Thus non-smooth optimization algorithms such as subgradient and bundle methods (see Outrata et al. (1998)) should be recommended to solve (10), in general. Nevertheless for the particular objective function f defined in problem (10) the gradient of f, $\nabla_x f$, exists and is defined by

$$\nabla_x f(x) = \nabla_x F(x)^T u(x) - \frac{1}{2} u(x)^T \nabla_x A(x) u(x). \tag{11}$$

For each x, the vector $u(x)$ is the solution of the complementarity problem and $\nabla_x F$ and $\nabla_x A$ are the gradients of F and A defined by

$$\nabla_x F(x) = (\nabla_x F_i(x))_{i \in H}, \quad \nabla_x A(x) = (\nabla_x A_{ij}(x))_{i,j \in H} \tag{12}$$

with F_i and A_{ij} the elements of F and A respectively. Assuming that F and A are of class C^1 with respect to x, the gradient $\nabla_x f$ is of class C^0. We observe that formula (11) can be obtained by calculating the directional derivative $f'(x, \tilde{x})$ of f at the point x in the direction $\tilde{x}$. In fact it follows from the definition of f and the complementarity problem (6) that

$$f'(x,\tilde{x}) = u(x)^T A(x) u'(x,\tilde{x}) + \frac{1}{2} u(x)^T A'(x,\tilde{x}) u(x) \tag{13}$$

where $A'(x,\tilde{x})$ is the directional derivative of f at x in the direction $\tilde{x}$ and $u'(x,\tilde{x})$ satisfies

$$\begin{array}{l} A(x)u'(x,\tilde{x}) = w'(x,\tilde{x}) + F'(x,\tilde{x}) - A'(x,\tilde{x})u(x) \\ u(x)^T w'(x,\tilde{x}) = 0 \end{array} \tag{14}$$

with $w'(x,\tilde{x})$ and $F^{'}(x,\tilde{x})$ the directional derivatives of w and F at x in the direction $\tilde{x}$, respectively. Introducing (14) in (13), the term $u'(x,\tilde{x})$ disappears. Since A and F are of class C^1, then $A'(x,\tilde{x}) = \nabla_x A \cdot \tilde{x}$ and $F'(x,\tilde{x}) = \nabla_x F \cdot \tilde{x}$, where the dot means the usual euclidean product in R^s. It is now easy to obtain the expression (11).

Therefore for the specific problem (10) it is possible to apply a classical projected gradient method. The steps of this algorithm are described next, where P_X represents the projection on the set X.

Projected-Gradient Algorithm

- Let $x^0 \in X$ and $\epsilon > 0$ be a given tolerance.
- For $k = 0, 1, 2, ...$
  - Compute $\nabla_x f(x^k)$, $y^k = P_X(x^k - \nabla_x f(x^k))$ and $p^k = y^k - x^k$.
  - If $\|p^k\| < \epsilon$, stop with $(x^k, u(x^k))$ a solution of the MPEC.
  - Compute the stepsize $\alpha_k \in]0,1]$ using the Armijo Criterion

    $$f(x^k + \alpha_k p^k) \leq f(x^k) + c\,\alpha_k \nabla_x f(x^k)^T p^k \tag{15}$$

    with $0 < c < 1$.
- Update $x^{k+1} = x^k + \alpha_k p^k$.

As discussed in Nocedal et al. (1999) this algorithm possesses global convergence to a stationary point of the function $f(x)$ on the convex set X. In order to compute objective function and gradient values and to employ the Armijo criterion, a complementarity algorithm is required to evaluate $u(x^k)$ and $u(x^k + \alpha_k p^k)$, that are the solutions of the complementarity problem (6) for $x = x^k$ and $x = x^k + \alpha_k p^k$, respectively. Since the matrix $A(x)$ is symmetric positive definite for each $x \in X$, the block pivoting or interior-point algorithms should be recommended to process these LCPs (see Fernandes et al. (2002)). Furthermore for this problem the projection is quite simple to obtain because the set X consists of simple bounds on the variables x_i.

4. A Numerical Example

We have tested the previous algorithms in an example. The material is an unidirectional fiber reinforced composite material, whose longitudinal modulus is $E = E_f V_f + E_m(1 - V_f)$ with E_f the modulus of the fiber, E_m the modulus of the matrix and V_f the fiber volume fraction which belongs to $[0, 1]$. The optimization variable is V_f and the data of the problem are displayed in the table below.

Table 1. Data of the Example

Parameter	*Value*
E_m (GPa) – modulus of the matrix	3.45
E_f (GPa) – modulus of the fiber	86
$V_f^{\min}$, $V_f^{\max}$ – lower and upper bounds for V_f	0.01, 0.99
L (m) – length of the rod	10
q (N) – distributed force in the direction of the rod axis	−18000
p (N) – distributed force in the direction perpendicular to the rod axis	−200
g (m) – obstacle (constant)	−0.001
$\|w\|$ (m^2) – area of the cross section	0.004
$I(m^4)$ – moment of inertia	$2.1e^{-6}$

The symbols (GPa), (N) and (m) denote the units Giga Pascal, Newton and meter, respectively. The interval $[0, 10]$ has been discretized with 50 finite elements, whose length h_i is constant and equal to 0.2.

The projected-gradient algorithm has successfully found an optimal solution $V_f = 0.3518$ in 7 iterations. We have employed the block pivoting algorithm described in Fernandes et al. (2002) to process the LCP's required by the projected-gradient method. The block pivoting algorithm has required a total of 150 iterations to process all the 44 LCP's needed by the projected-gradient algorithm. In these tests the subroutine *beam2e* of the CALFEM toolbox of MATLAB has been used to evaluate the stiffness matrix A and the force vector F. The complementarity algorithm has been implemented in MATLAB. The figure 1 shows the plot of the equilibrium bending displacement of the rod, at the optimum value $V_f = 0.3518$.

As stated in this section, the projected gradient algorithm has performed well for solving this example of MPEC problem discussed in this paper. Further computational investigation is required to evaluate the performance of the algorithm in practice. A description of this experience will be reported in the near future.

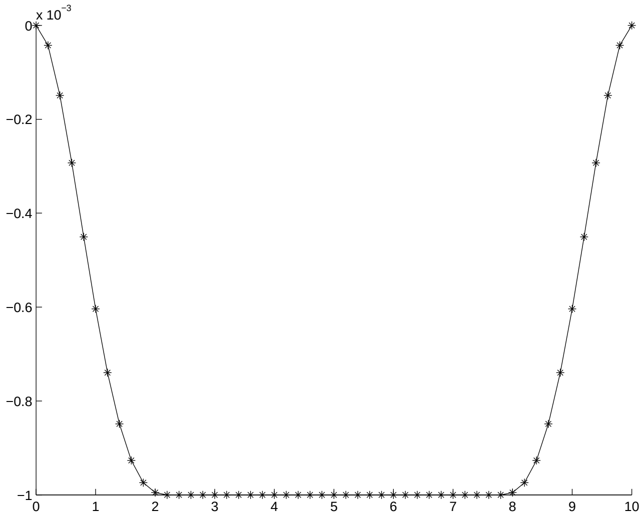

Figure 1. Bending displacement of the rod for $V_f = 0.3518$

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