Complementarity and genetic algorithms for an optimization shell problem

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Abstract

In this paper an optimization thin laminated shallow shell problem is discussed. The existence of a solution for both the linear and nonlinear versions of this problem is firstly studied by exploiting their reductions into variational inequalities. The discretization of these continuous problems by using appropriate finite elements leads into a Mathematical Programming Problem with Equilibrium Constraints (MPEC) in which some of their variables assume integer values and the remaining variables are implicitly defined as the solution of a Mixed Complementarity Problem (MCP). A genetic algorithm incorporating a complementarity path-following technique is proposed for the solution of this MPEC. It is shown that the efficiency of this hybrid problem depends on the problem to be linear or nonlinear. Some computational experience with this algorithm on the solution of special cases of this MPEC has been reported elsewhere and is briefly described to highlight the performance of the proposed methodology.

Key words: Shell Problems, Variational Inequalities, MPEC, Complementarity Problems, Genetic Algorithms

1 Introduction

Let S be a thin elastic laminated shallow shell, made of $2n$ laminas which are symmetrical disposed, both from a material and a geometric properties standpoint, with respect to the middle surface of the shell. Each lamina i, for $i = 1, ..., n$, is supposed to be made of a material m_i, having a monoclinic behaviour through the thickness of the laminate. The thickness t_i of lamina i is defined by $t_i = h_i - h_{i-1}$, where h_i is the distance, measured along the direction of the unit normal vector to the middle surface, from the middle surface to the upper face of lamina i. In addition, the shell is subjected to a vertical load, clamped on the boundary, and the vertical displacement of the middle surface is constrained by an obstacle. Finally, upper bounds on the global cost, weight and thickness of the laminated shell are also imposed.

Given discrete sets of materials $M = \{m_j : j = 1, 2, 3, ..., q \ (q > n)\}$, of thickness $T = \{t_k : k = 1, 2, 3, ..., p \ (p > n)\}$ and of functions $\Phi = \{\vec{\phi}_l : l = 1, 2, 3, ..., r \ (r > 2)\}$ defining the middle surface of the shell, the objective is to select the materials and thickness, m_i and t_i of each lamina i, and a function $\vec{\phi}$, in order to minimize the strain energy of the two-dimensional laminated shallow shell model.

The variational formulation of this problem takes the following form

$$\left\{ \begin{array}{l} \min\limits_{s \in C} F(s, \vec{u}^s) \\ \text{subject to:} \quad \left\{ \begin{array}{l} \vec{u}^s \in V \\ \Pi^s(\vec{u^s}) = \min\limits_{\vec{v} \in V} \Pi^s(\vec{v}). \end{array} \right. \end{array} \right. \tag{1}$$

The vector of optimization variables is defined by $s = (s_M, s_T, s_\Phi)$, where s_M is a vector of materials with components in M, s_T is a vector of thickness with components in T and s_Φ is a vector with only one component, belonging to the set Φ. Hence the variables s_i should assume integer values. The set V contains the admissible displacements of the middle surface; Π^s and F are, respectively, the total potential energy and the strain energy of the laminated shell. The set C is a subset of $M \times T \times \Phi$, that imposes constraints as the global cost, weight and thickness of the laminated shell.

The main goal of this paper is to describe and to investigate the properties of the bilevel problem (1) and its numerical solution. The inner optimization problem in (1) can be reformulated as a variational inequality, that is more useful for its numerical solution by the finite element method. A hybrid algorithm is proposed that combines complementarity path-following techniques [7], [19] for solving the variational inequality, with a genetic algorithm [10] for the minimization of the functional. Two important special instances of problem (1) have been discussed elsewhere, namely the obstacle problem for an elastic plate [8] and the compliance minimization of a composite laminated plate [4]. The results of the experiences on the solution of these two problems by the proposed methodology are also reviewed in this paper.

The rest of the paper is organized as follows: in section 2 and 3 notations and hypotheses on the geometry, the material properties of the shell, the expression of the strain and curvature tensors and the exact definitions of V, Π^s and F are introduced. Equivalent formulations and properties of problem (1) and its discrete formulation are discussed in the next two sections. In section 6 the proposed hybrid algorithm for the solution of the discrete problem is described together with its application for the solution of some of their special cases. Finally some conclusions are stated in the last part of the paper.

2 Notations and Hypotheses

In this section, we introduce some notations and hypotheses that will prove useful for the definitions of V, Π^s and F of problem (1).

As far as the notations are concerned, greek indices or exponents α, β, μ,... belong to the set $\{1,2\}$ and the latin indices or exponents i, j, k,... belong to the set $\{1,2,3\}$. The summation convention with respect to repeated indices and exponents is used; the euclidean scalar and vector product of two vectors $\vec{u}$ and $\vec{v}$ in R^3 are denoted by $\vec{u}\cdot\vec{v}$ and $\vec{u}\times\vec{v}$ respectively, and $\|.\|$ denotes the euclidean norm.

The hypotheses of the models are discussed next and are concerned with the geometry and material properties of the shell, the strain and curvature tensors of the middle surface.

2.1 Geometry of the shell

The middle surface $\overline{\Omega}\subset R^3$ of the shell is the image of an open, connected and bounded subset $\omega\subset R^2$, by a sufficiently smooth mapping $\vec{\phi}$.

The covariant and the contravariant basis, $(\vec{a}_\alpha)$ and $(\vec{a}^\beta)$, of the tangent plane of the middle surface are defined by $\vec{a}_\alpha=\vec{\phi}_{,\alpha}$ and $\vec{a}^\beta\cdot\vec{a}_\alpha=\delta^\beta_\alpha$, where δ^β_α is the Kronecker's symbol, that is, $\delta^\beta_\alpha=1$, if $\alpha=\beta$ and $\delta^\beta_\alpha=0$, if $\alpha\neq\beta$ and $._{,\alpha}$ means the usual derivation with respect to the component ξ^α of the variable $\xi=(\xi^1,\xi^2)$ in ω.

The unit normal vector is $\vec{a}_3=\vec{a}^3=\frac{\vec{a}_1\times\vec{a}_2}{\|\vec{a}_1\times\vec{a}_2\|}$ and ξ^3 denotes the variable along the vertical axis with the direction of $\vec{a}_3$.

A shell S with middle surface $\vec{\phi}(\overline{\omega})$ and constant thickness t is the set of points P in R^3 defined by

$$S=\{\vec{OP}:\quad \vec{OP}=\vec{\phi}(\xi^1,\xi^2)+\xi^3\vec{a}_3(\xi^1,\xi^2),\quad -\frac{t}{2}\leq\xi^3\leq\frac{t}{2}\}\tag{2}$$

where O is the origin of the reference system, $\vec{\phi}(\xi^1,\xi^2)$ represents the projection of $\vec{OP}$ in the middle surface and $|\xi^3|$ is the distance from P to its projection, measured along the direction of the unit normal vector $\vec{a}^3$.

The Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ and the covariant components $a_{\alpha\beta}$ and $b_{\alpha\beta}$ of the first and second fundamental forms of the middle surface are given by

$$\Gamma^\alpha_{\beta\gamma}=\vec{a}^\alpha\cdot\vec{a}_{\gamma,\beta}=\vec{a}^\alpha\cdot\vec{a}_{\beta,\gamma}=\Gamma^\alpha_{\gamma\beta},\quad a_{\alpha\beta}=\vec{a}_\alpha\cdot\vec{a}_\beta,\quad b_{\alpha\beta}=-\vec{a}_\alpha\cdot\vec{a}_{3,\beta}.\tag{3}$$

Furthermore $a=det(a_{\alpha\beta})=a_{11}a_{22}-a_{12}^2\neq 0$. The covariant derivatives of a vector field $\vec{v}$ defined on the middle surface are denoted by a vertical bar $|$, that is,

$$\begin{aligned}&v^\alpha_{|\mu}=v^\alpha_{,\mu}+\Gamma^\alpha_{\lambda\mu}v^\lambda,\quad v_{\alpha|\mu}=v_{\alpha,\mu}-\Gamma^\lambda_{\alpha\mu}v_\lambda,\quad v_{3|\alpha}=v_{3,\alpha},\\&v_{3|\alpha\beta}=v_{3,\alpha\beta}-\Gamma^\lambda_{\alpha\beta}v_{3,\lambda}\end{aligned}\tag{4}$$

where v^α and v_α are the contravariant and the covariant components of the vector field $\vec{v}$, respectively.

In particular a shallow shell is a shell which has a weak curvature, that is, a shell such that $b_{\alpha\beta}$ and $b_{\alpha\beta|\lambda}$ are very small when compared to the unity.

2.2 Material properties of the shell

Each lamina i is supposed to be made of an anisotropic and nonhomogeneous material, with elastic symmetry with respect to the surface ξ^3 = constant, that is, a monoclinic material whose elastic coefficients C_i^{jklm}, for each lamina i, satisfy [11], [17]

$$\begin{cases} C_i^{jklm} = C_i^{kjlm} = C_i^{jkml} = C_i^{mljk}, \\ C_i^{\alpha\beta\lambda 3} = C_i^{\alpha 333} = 0, \\ \exists c > 0 : \; C_i^{jklm}\tau_{jk}\tau_{lm} \geq c \displaystyle\sum_{j,k=1}^{3} |\tau_{jk}|^2, \; \forall(\tau_{jk}) \text{ symmetric tensor.} \end{cases} \tag{5}$$

2.3 Strain and curvature tensors of the middle surface

Two thin elastic shallow shell models are adopted. The expressions of the covariant components $\gamma_{\alpha\beta}(.)$ and $\rho_{\alpha\beta}(.)$ of the strain tensor and the change of curvature tensor of the middle surface are given by

$$\gamma_{\alpha\beta}(\vec{v}) = \frac{1}{2}(v_{\alpha|\beta} + v_{\beta|\alpha}) - b_{\alpha\beta}v_3, \qquad \rho_{\alpha\beta}(\vec{v}) = v_{3|\alpha\beta}, \tag{6}$$

for the linear case [2], and by

$$\gamma_{\alpha\beta}(\vec{v}) = \frac{1}{2}(v_{\alpha|\beta} + v_{\beta|\alpha}) - b_{\alpha\beta}v_3 + \frac{1}{2}v_{3,\alpha}v_{3,\beta}, \qquad \rho_{\alpha\beta}(\vec{v}) = v_{3|\alpha\beta}, \tag{7}$$

for the nonlinear case [1], [14].

3 Definition of V, Π^s and F

The set of admissible displacements $\vec{u}$ of the middle surface of the shell is defined by

$$V = \{\vec{u} \equiv (u_1, u_2, u_3) \equiv (\underline{u}, u_3) \in [H_0^1(\omega)]^2 \times K\} \tag{8}$$

where

$$K = \{z \in H_0^2(\omega) : \quad z(\xi^1, \xi^2) \geq \psi(\xi^1, \xi^2), \quad a.e. \text{ in } \omega\}. \tag{9}$$

Here $\psi \leq 0$ is the function representing the obstacle and $H_0^1(\omega)$, $H_0^2(\omega)$ are Sobolev spaces defined by

$$\begin{aligned} H_0^1(\omega) &= \{v \in H^1(\omega) : \; v_{|\partial\omega} = 0\}, \\ H_0^2(\omega) &= \{v \in H^2(\omega) : \; v_{|\partial\omega} = \tfrac{\partial v}{\partial n} = 0\}, \end{aligned} \tag{10}$$

with $\partial\omega$ the boundary of ω and $\frac{\partial}{\partial n}$ the normal derivative.

The functional $\Pi^s(\vec{v})$ is the total potential energy of the shell given by

$$\Pi^s(\vec{v}) = \frac{1}{2} b^s(\vec{v}, \vec{v}) - L(\vec{v}). \tag{11}$$

The form $L(.)$ is a linear scalar form in V, related to the vertical force, acting on the shell, defined by

$$L(\vec{v}) = \int_\omega f^3\, v_3 \sqrt{a}\, d\xi^1\, d\xi^2, \tag{12}$$

where $f^3 \in L^2(\omega)$ is the given intensity of vertical force. The form $\frac{1}{2} b^s(\vec{v}, \vec{v})$ is the strain energy of the laminated shallow shell and its expression is

$$\left\{ \begin{array}{l} b^s(\vec{v}, \vec{v}) = 2 \displaystyle\sum_{i=1}^{n} \int_\omega \Bigg[\Big(\int_{h_{i-1}}^{h_i} A_i^{\alpha\beta\lambda\mu}\, d\xi^3 \Big)\, \gamma_{\alpha\beta}(\vec{v})\, \gamma_{\lambda\mu}(\vec{v}) + \\ \qquad \Big(\int_{h_{i-1}}^{h_i} (\xi^3)^2 A_i^{\alpha\beta\lambda\mu}\, d\xi^3 \Big)\, \rho_{\alpha\beta}(\vec{v})\, \rho_{\lambda\mu}(\vec{v}) \Bigg] \sqrt{a}\, d\xi^1 d\xi^2, \end{array} \right. \tag{13}$$

where $\gamma_{\alpha\beta}(.)$ and $\rho_{\alpha\beta}(.)$ are, respectively, the covariant components of the strain tensor and the change of curvature tensor of the middle surface, of the linear or nonlinear model (6) or (7), and $A_i^{\alpha\beta\lambda\mu}$ are the reduced elasticity coefficients, of lamina i, defined by

$$A_i^{\alpha\beta\lambda\mu} = C_i^{\alpha\beta\lambda\mu} - \frac{C_i^{\alpha\beta 33}\, C_i^{33\lambda\mu}}{C_i^{3333}}. \tag{14}$$

An explanation of formula (13) is given in [3], by using the asymptotic development technique, with the half-thickness of the laminate as a small parameter. The expression (13) can also be obtained directly from the formula of the strain energy of the three dimensional shell model, that is, from the computation of the integral

$$\int_S \sigma^{kj}\, \varepsilon_{kj} \tag{15}$$

where $\sigma^{kj} = C^{kjlm}\, \varepsilon_{lm}$ are the components of the three dimensional stress tensor, ε_{lm} are the components of the three dimensional strain tensor and C^{kjlm} are the elastic coefficients of the laminate, and where it is assumed that $\varepsilon_{\alpha\beta} = \gamma_{\alpha\beta} + \xi^3 \rho_{\alpha\beta}$ ($\gamma_{\alpha\beta}$, $\rho_{\alpha\beta}$ are given by (6) or (7)), $\varepsilon_{\alpha 3} = 0$, $\sigma^{33} = 0$ and $C^{kjlm} = C_i^{kjlm}$ in lamina i. Moreover, cross products of the type $\gamma_{\alpha\beta}(\vec{v})\, \rho_{\alpha\beta}(\vec{v})$ do not appear in (13), because the laminas are symmetrical with respect to the middle surface of the shell.

Finally the objective functional in (1) is defined by

$$F(s, \vec{u}^s) = \frac{1}{2} b^s(\vec{u}^s, \vec{u}^s). \tag{16}$$

4 Variational inequality formulation of the inner problem

In this section the inner mathematical problem

$$\left\{\begin{array}{l} \vec{u}^s \in V \\ \Pi^s(\vec{u^s}) = \min\limits_{\vec{v}\in V} \Pi^s(\vec{v}) \end{array}\right. \tag{17}$$

is briefly discussed. This problem represents the constraint set of problem (1) and its function depends on the problem to be linear or nonlinear.

For a fixed s, the solution of (17) is the triple composed by the covariant components (u_1^s, u_2^s, u_3^s) of the displacement $\sum_{i=1}^3 u_i^s\,\vec{a}^i$ of the points of the middle surface $\vec{\phi}(\overline{\omega})$ of the shell, when it is subjected to the action of a vertical force, and the normal displacement $u_3^s\,\vec{a}^3$ is constrained by the obstacle ψ.

As the function Π^s is Gâteaux differentiable in $[H_0^1(\omega)]^2 \times H_0^2(\omega)$, problem (17) is equivalent to the following variational inequality [12]

$$\left\{\begin{array}{l} \vec{u}^s \in V \\ < D\Pi^s(\vec{u^s}), \vec{v} - \vec{u^s} > \geq 0, \quad \forall \vec{v} \in V \end{array}\right. \tag{18}$$

where $< D\Pi^s(\vec{u^s}), \vec{v} >$ is the Gâteaux derivative of Π^s at $\vec{u^s}$ in the direction $\vec{v}$.

As $V = [H_0^1(\omega)]^2 \times K$, choosing in (18) $\vec{v} = (0, v_3)$ and subsequently $\vec{v} = (\underline{v} + \underline{u}, 0)$ and $\vec{v} = (-\underline{v} + \underline{u}, 0)$ it is easy to show that the variational inequality (18) is equivalent to a system composed of another variational inequality and an equation. The expressions of these systems for the linear and nonlinear cases are stated below.

- Linear case

$$\left\{\begin{array}{l} \text{Find } \vec{u}^s \equiv (u_1, u_2, u_3) \equiv (\underline{u}, u_3) \in V, \text{ such that} \\ A^s(u_3, v_3 - u_3) + a^s(\underline{u}, v_3 - u_3) - L(v_3 - u_3) \geq 0, \quad \forall v_3 \in K \\ B^s(\underline{u}, \underline{v}) + c^s(u_3, \underline{v}) = 0, \quad \forall \underline{v} \in [H_0^1(\omega)]^2. \end{array}\right. \tag{19}$$

- Nonlinear case

$$\left\{\begin{array}{l} \text{Find } \vec{u}^s \equiv (u_1, u_2, u_3) \equiv (\underline{u}, u_3) \in V, \text{ such that} \\ A^s(u_3, v_3 - u_3) + a^s(\underline{u}, u_3; v_3 - u_3) - L(v_3 - u_3) \geq 0, \quad \forall v_3 \in K \\ B^s(\underline{u}, \underline{v}) + d^s(u_3, \underline{v}) = 0, \quad \forall \underline{v} \in [H_0^1(\omega)]^2. \end{array}\right. \tag{20}$$

Furthermore the definitions of the forms in these problems are presented next:

$$\left\{\begin{array}{l} A^s(u_3, v_3) = 2\sum_{i=1}^n \int_\omega \Bigg[\Big(\int_{h_{i-1}}^{h_i} A_i^{\alpha\beta\lambda\mu}\, d\xi^3 \Big)\, b_{\alpha\beta} u_3\, b_{\lambda\mu} v_3 + \\ \qquad \Big(\int_{h_{i-1}}^{h_i} (\xi^3)^2 A_i^{\alpha\beta\lambda\mu}\, d\xi^3 \Big)\, u_{3|\alpha\beta}\, v_{3|\lambda\mu} \Bigg] \sqrt{a}\, d\xi^1 d\xi^2, \end{array}\right. \tag{21}$$

$$
\left\{
\begin{aligned}
B^s(\underline{u},\underline{v}) = 2\sum_{i=1}^{n}\int_\omega \Bigg[\Big(\int_{h_{i-1}}^{h_i} A_i^{\alpha\beta\lambda\mu}\,d\xi^3\Big).\\
\tfrac{1}{4}(u_{\alpha|\beta}+u_{\beta|\alpha})\,(v_{\lambda|\mu}+v_{\mu|\lambda})\Bigg]\sqrt{a}\,d\xi^1 d\xi^2,
\end{aligned}
\right.
\tag{22}
$$

$$
c^s(u_3,\underline{v}) = -2\sum_{i=1}^{n}\int_\omega \Bigg[\Big(\int_{h_{i-1}}^{h_i} A_i^{\alpha\beta\lambda\mu}\,d\xi^3\Big)\, b_{\alpha\beta}u_3\,\frac{1}{2}\,(v_{\lambda|\mu}+v_{\mu|\lambda})\Bigg]\sqrt{a}\,d\xi^1 d\xi^2,
\tag{23}
$$

$$
\left\{
\begin{aligned}
d^s(u_3,\underline{v}) = 2\sum_{i=1}^{n}\int_\omega \Bigg[\Big(\int_{h_{i-1}}^{h_i} A_i^{\alpha\beta\lambda\mu}\,d\xi^3\Big)\,(-b_{\alpha\beta}u_3+\\
\tfrac{1}{2}u_{3,\alpha}u_{3,\beta})\tfrac{1}{2}(v_{\lambda|\mu}+v_{\mu|\lambda})\Bigg]\sqrt{a}\,d\xi^1 d\xi^2,
\end{aligned}
\right.
\tag{24}
$$

$$
\left\{
\begin{aligned}
a^s(\underline{u},u_3;v_3) = 2\sum_{i=1}^{n}\int_\omega \Bigg[\Big(\int_{h_{i-1}}^{h_i} A_i^{\alpha\beta\lambda\mu}\,d\xi^3\Big)\,\Big(-b_{\alpha\beta}u_3\,u_{3,\mu}\,v_{3,\lambda}+\\
\tfrac{1}{2}[u_{\alpha|\beta}+u_{\beta|\alpha}+u_{3,\alpha}u_{3,\beta}][u_{3,\mu}v_{3,\lambda}-b_{\lambda\mu}v_3]\Big)\Bigg]\sqrt{a}\,d\xi^1 d\xi^2,
\end{aligned}
\right.
\tag{25}
$$

$$
a^s(\underline{u},v_3) = c^s(v_3,\underline{u}),
\tag{26}
$$

$$
L(v_3) = \int_\omega f^3\, v_3\sqrt{a}\,d\xi^1 d\xi^2.
\tag{27}
$$

It is also worthwhile to mention that for the linear model

$$
< D\Pi^s(\vec{u^s}),\vec{v} >= b^s(\vec{u^s},\vec{v}) - L(\vec{v}),
\tag{28}
$$

so the variational inequality (18) or the system (19) are equivalent to

$$
\left\{
\begin{array}{l}
\vec{u}^s \in V\\
b^s(\vec{u^s},\vec{v}-\vec{u^s}) - L(\vec{v}-\vec{u^s}) \geq 0, \quad \forall \vec{v}\in V.
\end{array}
\right.
\tag{29}
$$

The existence of a solution to the problem (17) depends on the properties of the operators and forms defining the systems (19) and (20). For the case where the laminate has only one ply and the material is homogeneous and isotropic, it is possible to establish two important existence results that are shown below.

- If the problem (17) is linear it is equivalent to the variational inequality (29). The bilinear form of (29) is elliptic if the mixed components b_α^β of the second form of the surface satisfy $|b_\alpha^\beta| \leq \epsilon$ and $|b_{\alpha|\lambda}^\beta| \leq \epsilon$, where $\epsilon > 0$ is some small real and positive number [2]. Hence by the Lions-Stampacchia theorem [15] the variational inequality has a unique solution and the same happens to problem (17).

- For the nonlinear case, the problem reduces to the variational inequality of (20) whose operator is nonlinear, pseudo-monotone and coercive, if the mixed components b_α^β of the second form of the surface satisfy $|b_\alpha^\beta| \leq \epsilon$ and $|b_{\alpha|\lambda}^\beta| \leq \epsilon$, where $\epsilon > 0$ is some small real and positive number [1]. Hence it also has at least a solution [15].

For the laminated linear or nonlinear shell problems (19) and (20), with more than one material, the same properties and results hold [9], using arguments similar to [1] and [2], because the reduced elastic coefficients $A_i^{\alpha\beta\gamma\mu}$ are smooth enough and satisfy the following symmetric and ellipticity conditions [9]

$$\begin{aligned} & A_i^{\alpha\beta\gamma\mu} = A_i^{\alpha\beta\mu\gamma} = A_i^{\mu\gamma\alpha\beta} = A_i^{\gamma\mu\alpha\beta}, \\ & \exists c > 0 : \ A_i^{\alpha\beta\gamma\mu}\tau_{\alpha\beta}\tau_{\gamma\mu} \geq c \sum_{\alpha,\beta=1}^{2} |\tau_{\alpha\beta}|^2, \ \forall(\tau_{\alpha\beta}) \text{ symmetric tensor.} \end{aligned} \tag{30}$$

5 Discrete Formulation

As is usual in the solution of these type of variational models, the finite element method is used to get a discrete problem that approximates the original continuous problem (1); see for instance [2], for the details of the use of this method in shell models. Due to the constraints involved in the continuous problem (1), the discrete problem takes the form of a Mathematical Program with Equilibrium Constraints (MPEC) [16]. In this section the definiton of the resulting MPEC is introduced.

Consider a finite element mesh of the domain ω, with m global degrees of freedom. Let L_1, L, H and I be four subsets of the index set $\{1, 2, 3, .., m\}$ such that:

- L_1 represents the indices of the degrees of freedom related to the vertical displacement u_3, at the nodes belonging to the interior of the mesh;

- L is a subset of L_1, corresponding to the degrees of freedom that represent the approximation of the vertical displacement u_3, at the interior nodes;

- I contains the indices of the degrees of freedom of the displacement $\vec{u}$ at the boundary nodes;

- H is the complement in $\{1, 2, 3, .., m\}$ of the set $L_1 \cup I$, that is, $H = \{1, 2, 3, .., m\} \setminus (I \cup L_1)$.

Let v be a vector in R^m, and denote by v_L, v_I, v_{L_1} or v_H the subvectors of v, whose components have indices belonging to L, I, L_1 or H respectively. Let $\psi_L = (\psi_i)_{i \in L}$ be the vector whose components are the values of the obstacle ψ at the nodes belonging to the set L, and K be the set that approximates the original set (9), that is

$$K = \{z \in R^m : \quad z_i \geq \psi_i, \quad i \in L\}. \tag{31}$$

Then, the discrete problem corresponding to (19) or (20) takes the following form

$$\begin{cases} \text{Find } u \in R^m, \text{ such that} \\ u_I = 0, \quad u_L \geq \psi_L \\ (z_{L_1} - u_{L_1})^T G^s_{L_1}(u) \geq 0 \\ z \in R^m, \quad z_L \geq \psi_L \\ G^s_H(u) = 0, \end{cases} \tag{32}$$

where u is the finite element approximation of $\vec{u}^s$, depending on s, and $G^s_{L_1}(u) = (G^s_i(u))_{i \in L_1}$ and $G^s_H(u) = (G^s_j(u))_{j \in H}$ are the functions obtained from the finite element discretization of the variational inequality and the equation, respectively, of systems (19) or (20). These functions are affine or nonlinear, depending on the continuous problem to be linear or nonlinear.

If $J = \{1, 2, 3, .., m\} \setminus (I \cup L)$ and $n = m - |I|$, where $|I|$ denotes the number of elements of the set I, then the variational inequality (32) is equivalent [5] to the following Mixed Complementarity Problem (MCP):

$$\begin{cases} \text{Find } u = (u_J, u_L) \in R^n, \text{ such that} \\ G^s_J(u) = 0, \\ G^s_L(u) \geq 0, \\ u_L \geq \psi_L, \\ (u_i - \psi_i) G^s_i(u) = 0, \quad \forall i \in L. \end{cases} \tag{33}$$

Note that the vector $u_I = 0$ has been eliminated from further consideration as its components should be equal to zero in any solution of the variational inequality (32).

If $F(s, u)$ is the finite element approximation of (16), then the discrete formulation of (1) is reduced to the following Mathematical Programming Problem with Equilibrium Constraints (MPEC):

$$\begin{cases} \min F(s, u) \\ \text{subject to:} \begin{cases} s \in C \\ u = u(s) \quad \text{is a solution of MCP (33).} \end{cases} \end{cases} \tag{34}$$

It is also important to add that all the variables s_i should assume integer values for the vector s to belong to the set C.

6 An Algorithm for the MPEC problem

It follows from the definition of the MPEC introduced in the previous section that the variables s_i should take integer values and the inner variables u_i depend implicitly on the outer variables s_i by means of a complementarity problem. These two features of the MPEC recommend the use of a genetic algorithm for its solution. This procedure works solely on the integer variables s_i. In order to compute the value of the objective function for a particular value $\bar{s}$ of s, the MCP (33) with $s = \bar{s}$ is first solved to get the variable $\bar{u} = u(\bar{s})$. Then the value is given by $F(\bar{s}, \bar{u})$.

Before describing the genetic algorithm for the solution of the MPEC (34), it is necessary to explain how the MCP (33) can be processed. Next, two path-following algorithms are discussed for this purpose, namely an interior-point method [13], [19] and the so-called PATH algorithm [7], [18].

6.1 An Interior Point Algorithm

The MCP (33) can be written as follows

$$G^s(u) - w = 0 \tag{35}$$

$$(U_L - \Psi_L)W_L\, e_L = 0 \tag{36}$$

$$w_J = 0 \tag{37}$$

$$u_L \geq \psi_L, \quad w_L \geq 0 \tag{38}$$

where $u,\ w \in R^n$, U_L, Ψ_L and W_L are diagonal matrices with diagonal elements equal to u_i, ψ_i and w_i, $i \in L$, repectively, e_L is a vector with $|L|$ components equal to one and $G^s(u) = (G^s_J(u), G^s_L(u))$.

The interior-point algorithm is an iterative procedure that seeks a solution of the system of nonlinear equations (35)-(36), by maintaining the requirements (37)-(38) in each iteration. The search for such a solution is done in the interior of the set defined by the constraints (38). To describe an iteration of this algorithm, let (u^k, w^k) be the current iterate satisfying

$$u^k_L > \psi_L, \quad w^k_L > 0, \quad w^k_J = 0. \tag{39}$$

As is discussed in [19], the so-called central parameter μ_k is firstly computed. This parameter is used to define a central path that the algorithm should move closely in order to avoid premature approximation to the boundary of the set defined by the constraints (38). The value of μ_k is given by

$$\mu_k = \frac{\nu}{|L|} \sum_{i \in L} (u^k_i - \psi_i) w^k_i \tag{40}$$

where $0 < \nu < 1$ is a fixed real number and $|L|$ is the number of elements of the set L.

The search direction Δ^k is then found as the Newton's direction for the nonlinear system defined by the equations (35) and (37) and the central path equation

$$(U_L - \Psi_L) W_L \, e_L = \mu_k \, e_L. \tag{41}$$

Hence $\Delta^k = (\Delta u^k, \Delta w^k)$ should satisfy the following system of linear equations:

$$\left\{ \begin{array}{l} \left[\begin{array}{ccc} \nabla_{JJ} G^s(u^k) & \nabla_{JL} G^s(u^k) & 0 \\ \nabla_{LJ} G^s(u^k) & \nabla_{LL} G^s(u^k) & -I_L \\ 0 & W_L^k & U_L^k - \Psi_L \end{array} \right] \left[\begin{array}{c} \Delta u_J^k \\ \Delta u_L^k \\ \Delta w_L^k \end{array} \right] = \\ \\ \left[\begin{array}{c} w_J^k - G_J^s(u^k) \\ w_L^k - G_L^s(u^k) \\ \mu_k e_L - (U_L^k - \Psi_L) W_L^k e_L \end{array} \right] \end{array} \right. \tag{42}$$

$$\Delta w_J^k = 0$$

where I_L is the identity matrix of order $|L|$, U_L^k and W_L^k are diagonal matrices with diagonal elements equal to u_i^k and w_i^k, $i \in L$, respectively and

$$\nabla G^s(u^k) = \left[\begin{array}{cc} \nabla_{JJ} G^s(u^k) & \nabla_{JL} G^s(u^k) \\ \nabla_{LJ} G^s(u^k) & \nabla_{LL} G^s(u^k) \end{array} \right] \tag{43}$$

is the jacobian of G^s at u^k.

After computing the search direction $\Delta^k = (\Delta u^k, \Delta w^k)$, a stepsize α_k is found in such a way that the new iterate

$$u^{k+1} = u^k + \alpha_k \, \Delta u^k, \quad w^{k+1} = w^k + \alpha_k \, \Delta w^k \tag{44}$$

satisfies the constraints (39) with $k = k + 1$. It is easy to see that the following expression for α_k is sufficient for this purpose:

$$\alpha_k = \delta_k \min\{\theta_1^k, \theta_2^k\} \tag{45}$$

where

$$\begin{aligned} \theta_1^k &= \min \left\{ \tfrac{u_i^k - \psi_i}{-(\Delta u^k)_i} : \quad (\Delta u^k)_i < 0, \ i \in L \right\}, \\ \theta_2^k &= \min \left\{ \tfrac{w_i^k}{-(\Delta w^k)_i} : \quad (\Delta w^k)_i < 0, \ i \in L \right\}, \end{aligned} \tag{46}$$

for some $0 < \delta_k < 1$.

The next step to be performed is to check whether the new iterate (u^{k+1}, w^{k+1}) given by (44) is an approximate solution of the MCP. This is done by simply verifying if the two following conditions hold:

$$\| w^{k+1} - G^s(u^{k+1}) \| < \varepsilon_1, \qquad \sum_{i \in L} (u_i^{k+1} - \psi_i) w_i^{k+1} < \varepsilon_2 \tag{47}$$

for some positive tolerances ε_1 and ε_2. If this is not the case, a new iteration is performed with the new iterate (u^{k+1}, w^{k+1}).

It follows from this description, that the algorithm only works if the linear system (42) has a solution. A sufficient condition for this property to hold is the jacobian $\nabla G^s(u^k)$ to be Positive Semi-Definite and its principal submatrix $\nabla_{JJ} G^s(u^k)$ to be Positive Definite for each iterate u^k [13], [19]. As is discussed in [13] a simple modification of this interior-point algorithm possesses global convergence to the solution of the MCP (33) under this property of $\nabla G^s(u^k)$. Special care must be taken for the choice of the stepsize in order the new iterate not to move far from the central path. This implies the need of a more involved line-search procedure for computing the step size α_k [13], [19].

Unfortunately, for the nonlinear version of the shell problem the MCP under consideration does not satisfy this requirement concerning the positive semi-definitness of the jacobian $\nabla G^s(u^k)$. So the possible application of the interior-point algorithm for solving the MCP (33) should be studied experimentally. As reported in [8], the performance of the algorithm for a particular case of the nonlinear shell problem, that is the plate problem, has been quite disappointing. The algorithm has been unable to solve any one of the problems tested in this experience.

In case of the linear version, $G^s(u)$ is affine in u and its jacobian is a constant matrix M^s, that does not depend on the vector u. Furthermore, as the bilinear form of the shallow shell problem is symmetric and elliptic, this matrix is symmetric positive definite. As is discussed in [19], simple care on the computation of the stepsize α_k is sufficient to guarantee global convergence to the interior-point algorithm in this case. The use of the simple procedure defined by (45) and (46) with $\delta_k = 0.9995$ is usually recommended in practice. As reported in [8], this simple version of the interior-point algorithm has proven quite efficient for solving all the MCP test problems associated to the plate problem, that is a special case of the linear version of the shell problem.

6.2 The PATH Algorithm

As discussed in [7], [18] the MCP (33) can be reformulated as the following system of nonsmooth equations

$$G_B^s(x) = 0 \tag{48}$$

where G_B^s is the so-called normal map. This function is defined by

$$G_B^s(x) = G^s(\pi(x)) + x - \pi(x) \tag{49}$$

where $\pi(x)$ is the projection of x on the set

$$B = \{x \in R^n : \ x_i \geq \psi_i, \quad i \in L\}, \tag{50}$$

that is, $\pi_i(x)$ satisfies

$$\pi_i(x) = \begin{cases} x_i, & \text{if } i \in J \\ \max\{x_i, \psi_i\}, & \text{if } i \in L. \end{cases} \tag{51}$$

The equivalence between the MCP (33) and the system of normal equations (48) is stated below.

Property 1 -

1. *If $\bar{x}$ is a solution of the system (48), then*

$$\bar{w} = G^s(\pi(\bar{x})), \quad \bar{u} = \pi(\bar{x}) \tag{52}$$

is a solution of MCP (35)-(38).

2. *If $(\bar{u}, \bar{w})$ is a solution of the MCP (35)-(38), then*

$$\bar{x} = \bar{u} - \bar{w} \tag{53}$$

is a solution of the system (48).•

The PATH algorithm [7], [18] aims to find a solution of the system of normal map equations (48) by using a strategy similar to the so-called Damped Newton's method [6]. As the equation has a nonsmooth feature, the computation of the Newton's direction and the implementation of the damping strategy to find the new iterate become more complicated. Next, we describe how these two steps are performed. As in the differentiable case, the Newton's iterate x_N^k is found as a root of the first order approximation $A(x^k)$ of G_B^s at x^k. For each $x \in R^n$

$$A(x^k)(x) = G^s(\pi(x^k)) + \nabla G^s(\pi(x^k))(\pi(x) - \pi(x^k)) + x - \pi(x) \tag{54}$$

and x_N^k should satisfy

$$A(x^k)(x_N^k) = 0. \tag{55}$$

The definition (54) of the first-order approximation and the property 1 relating systems of normal map equations and complementarity problems, imply that x_N^k can be found by the following procedure:

- let

$$\begin{array}{l} M = \nabla G^s(\pi(x^k)) \\ q = G^s(\pi(x^k)) - M\pi(x^k), \end{array} \tag{56}$$

- solve the Mixed Linear Complementarity Problem (MLCP)

$$\begin{array}{l} y = q + Mv \\ v_i \geq \psi_i, \quad y_i \geq 0, \quad y_i(v_i - \psi_i) = 0, \quad i \in L \\ y_j = 0, \quad j \in J, \end{array} \tag{57}$$

- if (y^k, v^k) is a solution of this MLCP, then

$$x_N^k = v^k - y^k. \tag{58}$$

As is usual in a damped Newton's method, the new iterate x^{k+1} should be a point in the path joining x^k and x_N^k that guarantees a sufficient decrease for a certain merit function associated with the system of normal map equations. By choosing the natural merit function

$$\|G_B^s(x)\|^2 \tag{59}$$

then the computation of x^{k+1} reduces to find a value t belonging to the interval $[0,1]$ such that $x^{k+1} = x(t)$ satisfies the property

$$\|G_B^s(x(t))\| \leq (1-\nu t)\|G_B^s(x^k)\| \tag{60}$$

for some fixed value $0 < \nu < 1$. Note that $x(0) = x^k$ and $x(1) = x_N^k$. In order to compute this value of t and the corresponding vector $x(t)$ the following condition

$$(1-t)G_B^s(x^k) = A(x^k)(x(t)) \tag{61}$$

is enforced. By exploiting once more the property 1, it is possible to show that the values $(t, x(t))$ correspond to solutions of the following Parametric MLCP

$$\begin{array}{l} y = (q-r) + tr + Mv \\ v_i \geq \psi_i, \quad y_i \geq 0, \quad y_i(v_i - \psi_i) = 0, \quad i \in L \\ y_j = 0, \quad j \in J \end{array} \tag{62}$$

where $r = G_B^s(x^k)$. Since $x(0) = x^k$ and $x(1) = x_N^k$ then the computation of $x^{k+1} = x(t)$ consists of solving the Parametric MLCP and finding t and the corresponding solution $x(t) = v(t) - y(t)$ that satisfies the sufficient decrease condition (60).

If x^k and x_N^k correspond to Basic Feasible Solutions (BFS) of the Parametric MLCP then the computation of these pairs $(t, x(t))$ simply amounts to perform pivot steps similar to those of the well-known Lemke's method [5] for linear complementarity problems with t and r the cover variable and vector. Each BFS corresponds to a breaking point of the piecewise line joining the current iterate x^k and the Newton's iterate x_N^k. The algorithm chooses one of these breaking points (BFS of the Parametric MLCP) satisfying the sufficient decrease condition (60).

There are two main difficulties in the implementation of this procedure, that are stated below:

- The iterates x^k and x_N^k may not correspond to BFS of the Parametric MLCP.
- An unbounded ray may occur during the generation of such a path.

As is fully discussed in [7], [18] the algorithm PATH provides remedies that work in many cases, as well as efficient procedures to find the values $(t, x(t))$ according to monotone or nonmonotone line search schemes. The algorithm possesses global convergence to a root of the system of normal map equations under some restrictive hypotheses [7], [18].

Unfortunately it does not seem easy to verify whether these hypotheses hold or not for the MCP associated with the nonlinear version of the shallow shell problem. So there is no guarantee that the algorithm converges to a solution of this MCP. As for the interior-point method, some special instances of this problem have been solved by the PATH algorithm. The results of these experiences are reported in [8] and show that the PATH algorithm has been able to solve some of these test problems but not all.

As before, the linear version of the shallow shell problem leads into a Mixed Linear Complementarity Problem with a Positive Definite Matrix. Hence this problem is solved by finding the Newton's iterate of the PATH algorithm. Therefore this last algorithm reduces to a modification of Lemke's method and converges to the unique solution of the MLCP in a finite number of pivot steps. As is reported in [8] the algorithm performs quite well in practice and usually finds the unique solution of the MLCP in a reasonable number of pivot steps for all the test problems that have been considered in the experiments [8].

6.3 The Genetic Algorithm

Genetic algorithms are search and optimization algorithms that model the process of natural evolution. Their main disadvantage is that they require a great number of evaluations, although they do not request any derivative information.

A genetic algorithm for the MPEC problem requires five main steps that are discussed below:

Step 1 A coding technique, that assigns to each variable s a binary string, referred to as a chromosome.

To exemplify this coding technique, consider, for instance, that the laminated shell has 2×3 laminas, and there are 7 admissible materials $M=\{1,2,3,...,7\}$, 15 admissible thickness $T=\{1,2,3,...,15\}$ and 3 admissible functions, defining the middle surface of the shell, $\Phi=\{1,2,3\}$. A possible distribution of materials and thickness and a possible choice for the function $\vec{\phi}$, indicated by the vector s in (1), is

$$s=(s_M,s_T,s_\Phi)=(\underbrace{4,1,7}_{materials},\underbrace{3,11,15}_{thickness},\underbrace{2}_{function}) \tag{63}$$

Note that component i ($i=1,2,3$) of subvectors s_M and s_T coincide with the number of the lamina, that is, laminas 1, 2, 3 correspond to the materials 4, 1, 7 and the thickness 3, 11, 15, respectively.

By expressing these numbers (4,1,..) in the binary system, the following binary string represents the vector s

$$\underbrace{100\,001\,111}_{materials}\,\underbrace{0011\,1011\,1111}_{thickness}\,\underbrace{10}_{function} \tag{64}$$

This is called a chromosome. Thus, with this coding, each chromosome has a total of 23 bits, being 3 bits for each material, 4 bits for each thickness and 2 bits for the function.

Step 2 An initialization procedure, that is, a random set of initial points s (generated from the admissible cartesian set $M \times T \times \Phi$), which is the initial population of chromosomes.

This population of chromosomes is the set where the search for the mininum is performed, using the so-called genetic operators to be discussed in step 4.

Step 3 An evaluation objective function, which is the discretized strain energy function $F(s,u)$ of the shell with a penalized function, corresponding to the constraints defined in the set C of (1).

As stated before, in order to evaluate the objective function, for each chromosome s, it is first necessary to use the complementarity algorithm, to obtain the solution u of MCP (33). The computation of $F(s,u)$ is done by using these two quantities s and u.

Step 4 Genetic operators act on the chromosomes and generate successively new populations of chromosomes, from the original one, based on probabilistic rules. The most usual operators are crossover, mutation and reproduction [10], that are briefly explained below.

1. The crossover operator starts by randomly selecting two chromosomes s_1 and s_2, see (63)-(64), from the population; next, the bits between two randomly selected positions, along their common length, are swapped, and define two new chromosomes s_3 and s_4 in the search set. For example, if the bits between positions 6 and 18 in s_1 and s_2 are swapped, the new chromosomes s_3 and s_4 are defined by

$$s_1 = 100\,001 \quad \overbrace{111\,0011\,1011}^{positions\, 7-17\, of\, s_1} \quad 1111\,10 = (\overbrace{4,1,7}^{materials}, \overbrace{3,11,15}^{thickness}, \overbrace{2}^{function})$$

$$s_2 = 001\,011 \quad \underbrace{101\,0001\,0011}_{positions\, 7-17\, of\, s_2} \quad 1110\,01 = (\underbrace{1,3,5}_{materials}, \underbrace{1,3,14}_{thickness}, \underbrace{1}_{function})$$

$$s_3 = 100\,001 \quad \overbrace{101\,0001\,0011}^{positions\, 7-17\, of\, s_2} \quad 1111\,10 = (\overbrace{4,1,5}^{materials}, \overbrace{1,3,15}^{thickness}, \overbrace{2}^{function})$$

$$s_4 = 001\,011 \quad \underbrace{111\,0011\,1011}_{positions\, 7-17\, of\, s_1} \quad 1110\,01 = (\underbrace{1,3,7}_{materials}, \underbrace{3,11,14}_{thickness}, \underbrace{1}_{function})$$

which means that the material of lamina 3 and the thickness of laminas 1 and 2 also change.

2. The mutation operator randomly selects a position in the chromosome s_1 and changes the corresponding bit with a given probability, thus defining a new chromosome s_5. For example, if the position 11 in s_1 is selected, the bit 0 changes to 1, and the thickness of lamina 1 changes from 3 to 7. The new chromosome s_5 is

$$s_1 = 100\,001\,111\,0\underline{0}11\,1011\,1111\,10 = (\overbrace{4,1,7}^{materials}, \overbrace{3,11,15}^{thickness}, \overbrace{2}^{function})$$

$$s_5 = 100\,001\,111\,0\underline{1}11\,1011\,1111\,10 = (\underbrace{4,1,7}_{materials}, \underbrace{7,11,15}_{thickness}, \underbrace{2}_{function})$$

3. The reproduction operator defines the process by which the new generation is created from the previous one. The chromosomes in one generation are transferred into the next generation, with a probability according to the value of their objective function; thus, a higher proportion of the chromosomes with the best objective function values will be present in the next generation.

Step 5 A stopping criterium, that can be, for instance, a maximum number of generations of chromosomes.

The steps 1-5 present a summary of a genetic algorithm for the discrete optimization problem (1). For the details of implementation of this type of genetic algorithms see [10].

It is worthwhile to mention that this genetic algorithm has been applied to a special case of the linear version of problem (1) consisting of the compliance minimization of a linear, composite, laminated plate. The discrete optimization variables are the materials and the angle of orientation of the fibers, in each ply of the plate. The thickness of each ply is constant and a constraint on the global cost of the materials is imposed. In this special case the vertical displacement of the plate is not constrained by any obstacle. In mathematical terms this means that in the definition of the MCP (33) the set L is empty. Hence the MCP reduces to a system of linear equations and it is not necessary to apply any complementarity algorithm to get values for the objective function. For this plate problem the genetic algorithm has successfully identified, in each ply, the materials and the angles of orientation, corresponding to the minimum compliance of the plate. We recommend [4] for a report of some experiences with the genetic algorithm in this special case.

7 Conclusions

In this paper a linear and a nonlinear optimization laminated shallow shell models have been described and analysed. A hybrid genetic algorithm incorporating a complementarity path-following method has been proposed for the numerical solution of the resulting MPEC discrete problem. It has been shown that this type of procedure is successful for the linear version of the problem. However, the application of this methodology for the nonlinear version of the problem, when an obstacle is included in its definition seems to be more difficult and requires further research on the design of a more efficient technique for processing the inner complementarity problem of the MPEC problem.

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