Solution of a General Linear Complementarity Problem using smooth optimization and its application to bilinear programming and LCP *

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Abstract

This paper addresses a General Linear Complementarity Problem (GLCP) that has found applications in global optimization. It is shown that a solution of the GLCP can be computed by finding a stationary point of a differentiable function over a set defined by simple bounds on the variables.

The application of this result to the solution of bilinear programs and LCPs is discussed. Some computational evidence of its usefulness is included in the last part of the paper.

Keywords: Global optimization, linear complementarity problems, bilinear programming, box constrained optimization.

1 Introduction

The General Linear Complementarity Problem (GLCP) consists of finding vectors $z \in \mathbb{R}^n$ and $y \in \mathbb{R}^l$ such that

$$q + Mz + Ny \geq 0 \tag{1}$$

$$p + Rz + Sy \geq 0 \tag{2}$$

$$z \geq 0, \;\; y \geq 0, \;\; z^T(q + Mz + Ny) = 0 \tag{3}$$

where M, N, R and S are given matrices of orders $n \times n$, $n \times l$, $m \times n$ and $m \times l$ respectively and $q \in \mathbb{R}^n$, $p \in \mathbb{R}^m$ are given vectors. The GLCP has been studied by many authors as an ingredient for solving some optimization problems [16, 17, 18, 21, 22, 23, 24, 30, 32, 37]. This problem is a generalization of the well-known Linear Complementarity Problem (LCP)

$$w = q + Mz, \;\; z \geq 0, \;\; w \geq 0, \;\; z^T w = 0 \tag{4}$$

as it reduces to this latter problem when the variables y_i and the constraints (2) do not exist.

It is known that the LCP can be solved in polynomial-time when M is a positive semi-definite (PSD) matrix, that is, if M satisfies $x^T Mx \geq 0$ for all $x \in \mathbb{R}^n$ [26, 35]. On the other hand, it was shown in [25] that the GLCP is NP-hard when M is a PSD matrix. However, the GLCP can be solved in polynomial-time if M is a PSD matrix and $R = 0$ in its constraint (2) [25, 43]. In this paper we denote this latter type of problem by PGLCP. This PGLCP is an important tool for finding global minima of bilinear programming problems (BLP), as any BLP can be cast as the problem of minimizing a linear function on a PGLCP [21, 24].

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Recently, many authors have investigated the solution of linear and nonlinear complementarity problems by finding stationary points of differentiable and nondifferentiable functions under linear constraints [8, 9, 11, 12, 15, 31, 34, 41]. Among many results of this type, the following quadratic program has been associated with the LCP

$$\begin{array}{ll} \text{Min} & z^T(q+Mz) \\ \text{subject to} & q+Mz \geq 0 \\ & z \geq 0 \end{array} \tag{5}$$

It has been shown [8] that a solution of the LCP can be found by computing a stationary point of this quadratic program provided M is a Row Sufficient (RS) matrix, that is, if the following implication holds

$$x_i(M^Tx)_i \leq 0 \text{ for all } i=1,\ldots,n \;\Rightarrow\; x_i(M^Tx)_i = 0 \text{ for all } i=1,\ldots,n \tag{6}$$

This result has been extended to the GLCP under the same hypothesis on the matrix M and $R=0$ [25]. It is important to add that any PSD matrix is also RS, whence this result has applications on the solution of PGLCPs and bilinear programs.

Due to the large variety of efficient algorithms for nonlinear programs with simple bounds, there has been great effort on finding merit functions for which their stationary points on these simple sets lead into solutions of the LCP [9, 11, 15, 34]. In this paper we extend these results and introduce the following merit function for the GLCP with $R=0$

$$\phi(z,y,w,v) = ||w-q-Mz-Ny||^2 + ||v-p-Sy||^2 + (\sum_{i=1}^{n}(z_iw_i)^g)^h \tag{7}$$

where $||\ ||$ denotes the euclidean norm and $g \geq 1$, $h \geq 1$ are real numbers such that $g > 1$ if $h = 1$. We show that any stationary point of this function on the set defined by zero lower-bounds on the variables is a solution of the GLCP provided M is a RS matrix.

As stated before, any bilinear program can be reduced to the minimization of a linear function on a PGLCP [21, 24]. We denote this problem by MINPGLCP. Hence the result mentioned before seems to have important applications on the solution of bilinear programs. On the other hand, it has been shown recently that a LCP can be transformed into a bilinear program [2, 33]. By using this reduction, we prove that any LCP is equivalent to a PGLCP with a further condition on one of its variables.

As we explain later in this paper, we believe that these results may have important implications on the solution of bilinear programs and NP-hard LCPs. It seems possible that the combination of a local search method for finding a stationary point of the merit function (7) on the set defined by zero lower bounds on the variables together with some heuristic procedure to move from one stationary point to other, will be able to compute global minima of the bilinear program and solutions of the LCP in a reasonable amount of work.

The structure of the paper is as follows. In section 2 we establish our main result that associates the GLCP with stationary points of the function (7) on a set defined by zero lower bounds on the variables. The use of this result in bilinear programs and LCPs is discussed in sections 3 and 4. Some numerical experience with PGLCPs associated with bilinear programs and LCPs is presented in section 5. Finally, some conclusions are drawn in the last section of the paper.

2 PGLCP and stationary points of the merit function

Consider again the GLCP defined by (1), (2) and (3) with $R=0$. By introducing the slack variables w_i and v_i for the linear constraints (1) and (2), we can write the GLCP in the form

$$w = q + Mz + Ny \tag{8}$$

$$v = p + Sy \tag{9}$$

$$z \geq 0, \;\; y \geq 0, \;\; w \geq 0, \;\; v \geq 0 \tag{10}$$

$$z^Tw = 0 \tag{11}$$

Let $\mathcal{K}$ be the feasible set consisting of the linear constraints (8), (9) and (10). As stated before, consider the following nonlinear program

$$\begin{array}{lll} \text{(NLP)} & & \\ \text{Minimize} & f(z,y,w,v) = & \|w-q-Mz-Ny\|^2 + \|v-p-Sy\|^2 + (\sum_{i=1}^n (z_i w_i)^g)^h \\ \text{subject to} & & z,w,y,v \geq 0 \end{array} \tag{12}$$

where $\|\ \|$ denotes the euclidean norm and $g, h \geq 1$ are real numbers such that $g > 1$ if $h = 1$. Then we can establish the following property.

Theorem 1 *If $\mathcal{K} \neq \emptyset$ and M is a RS matrix, then any stationary point of NLP is a solution of the GLCP (8)–(11).*

Proof: If $(\bar{z}, \bar{w}, \bar{y}, \bar{v})$ is a stationary point of the NLP (12), then there exist Lagrange multipliers $\bar{\alpha} \in \mathbb{R}^n, \bar{\beta} \in \mathbb{R}^n, \bar{\gamma} \in \mathbb{R}^l$ and $\bar{\mu} \in \mathbb{R}^m$ such that $(\bar{z}, \bar{w}, \bar{y}, \bar{v}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\mu})$ satisfies the following conditions:

$$\alpha_k = 2[w-q-Mz-Ny]_k + gh[\sum_{i=1}^n (z_i w_i)^g]^{h-1} z_k^g w_k^{g-1}, \ \ k = 1, \ldots, n \tag{13}$$

$$\beta_k = -2[M^T(w-q-Mz-Ny)]_k + gh[\sum_{i=1}^n (z_i w_i)^g]^{h-1} z_k^{g-1} w_k^g, \ \ k = 1, \ldots, n \tag{14}$$

$$\gamma = -2N^T(w-q-Mz-Ny) - 2S^T(v-p-Sy) \tag{15}$$

$$\mu = 2(v-p-Sy) \tag{16}$$

$$z, w, y, v, \alpha, \beta, \gamma, \mu \geq 0 \tag{17}$$

$$\alpha_k w_k = z_k \beta_k = 0, \ \ k = 1, \ldots, n \tag{18}$$

$$y^T \gamma = \mu^T v = 0 \tag{19}$$

Let

$$\theta = w - q - Mz - Ny \in \mathbb{R}^n$$

$$\eta = gh[\sum_{i=1}^n (z_i w_i)^g]^{h-1} \in \mathbb{R}^1$$

Then by (13) and (14)

$$\theta_k = \frac{1}{2}[\alpha_k - \eta z_k^g w_k^{g-1}]$$

$$(M^T\theta)_k = -\frac{1}{2}[\beta_k - \eta z_k^{g-1} w_k^g]$$

for $k = 1, \ldots, n$. Hence for each $k = 1, \ldots, n$ we have

$$\theta_k (M^T\theta)_k = -\frac{1}{4}[\alpha_k \beta_k + \eta^2 (z_k w_k)^{2g-1} - \eta z_k^{g-1} w_k^{g-1} (w_k \alpha_k + z_k \beta_k)] \tag{20}$$

Now, by (18), $w_k \alpha_k + z_k \beta_k = 0$ for each $k = 1, \ldots, n$. Furthermore, all the variables are nonnegative by (17) and this implies

$$\theta_k (M^T\theta)_k \leq 0 \ \text{ for all } \ k = 1, \ldots, n.$$

Since M is a RS matrix then

$$\theta_k (M^T\theta)_k = 0 \ \text{ for all } \ k = 1, \ldots, n.$$

Then by (20), we have

$$\alpha_k \beta_k + \eta^2 (z_k w_k)^{2g-1} = 0$$

and

$$\alpha_k \beta_k = z_k w_k = 0 \ \text{ for all } \ k = 1, \ldots, n.$$

Since $g, h \geq 1$ and $g > 1$ if $h = 1$, then $z_k w_k = 0$ for all $k = 1, \ldots, n$, implies

$$[\sum_{i=1}^{n}(z_i w_i)^g]^{h-1} z_k^{g-1} w_k^{g-1} = 0 \text{ for all } k = 1, \ldots, n.$$

Hence (13) and (14) take the form

$$\alpha_k = 2[w - q - Mz - Ny]_k, \quad \beta_k = -2[M^T(w - q - Mz - Ny)]_k$$

Therefore the conditions (13)–(19) for a stationary point of the NLP can be rewritten as follows

$$\begin{array}{l} \alpha = 2(w - q - Mz - Ny) \\ \beta = -2M^T[w - q - Mz - Ny] \\ \gamma = -2N^T[w - q - Mz - Ny] - 2S^T[v - p - Sy] \\ \mu = 2(v - p - Sy) \\ z, w, y, v, \alpha, \beta, \gamma, \mu \geq 0 \\ \alpha^T w = z^T \beta = y^T \gamma = \mu^T v = 0 \end{array} \tag{21}$$

But these are the necessary and sufficient optimality conditions for the convex quadratic program

$$\begin{array}{ll} \text{Minimize} & ||w - q - Mz - Ny||^2 + ||v - p - Sy||^2 \\ \text{subject to} & w, y, z, v \geq 0 \end{array} \tag{22}$$

Since the constraint set of the GLCP is nonempty, this quadratic program has an optimal solution with value zero. Due to the equivalence between the conditions (21) and the optimal solution of the quadratic program (22), the stationary point $(\bar{z}, \bar{y}, \bar{w}, \bar{v})$ of NLP satisfies

$$w - q - Mz - Ny = 0$$

$$v - p - Sy = 0$$

$$w, z, y, v \geq 0$$

But we have shown before that $(\bar{z}, \bar{y}, \bar{w}, \bar{v})$ also satisfies $\bar{z}^T \bar{w} = 0$. So $(\bar{z}, \bar{y}, \bar{w}, \bar{v})$ is a solution of the GLCP and this proves the theorem. $\square$

It is important to add that the merit function (12) can be seen as an extension of two merit functions that have been discussed before for the LCP. In fact, by fixing $g = 2$ and $h = 1$ we get the so-called Natural Merit Function

$$\phi_1(z, w, y, v) = ||w - q - Mz - Ny||^2 + ||v - p - Sy||^2 + \sum_{i=1}^{n} z_i^2 w_i^2 \tag{23}$$

This function is an extension for the GLCP of the function

$$\phi_1(z, w) = ||w - q - Mz||^2 + \sum_{i=1}^{n} z_i^2 w_i^2$$

that has been used by many authors for the solution of the LCP [34, 40]. Theorem 1 implies that any stationary point of the program

$$\begin{array}{ll} \text{Minimize} & \phi_1(z, w) \\ \text{subject to} & z \geq 0, w \geq 0 \end{array}$$

is a solution of the LCP provided the LCP is feasible and M is RS matrix. This property extends for the RS matrices the results established in [34].

On the other hand, if $g = 1$, we get the function

$$\phi_2(z, w, y, v) = ||w - q - Mz - Ny||^2 + ||v - p - Sy||^2 + (z^T w)^h \tag{24}$$

that has been introduced in [15] for the LCP. Therefore theorem 1 extends for the GLCP the result presented in [15].

Since any PSD matrix is also a RS matrix, then the result mentioned in this section is also valid for the PGLCP. In section 4 we report some computational experience on the solution of PGLCPs with PSD matrices M by finding stationary points of the merit functions ϕ_1 and ϕ_2.

3 Application to Bilinear Programming

The Bilinear Programming Problem (BLP) is usually stated in the following form

$$\begin{array}{ll} \text{Minimize} & c^T x + d^T y + x^T H y \\ \text{subject to} & \;\; Ax \geq a \quad By \geq b \\ & \;\;\; x \geq 0 \quad y \geq 0 \end{array} \tag{25}$$

where $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$ and $H \in \mathbb{R}^{n_1 \times n_2}$ is in general a rectangular matrix. It follows from the definition of the BLP that the variables x and y belong to two disjoint constraint sets

$$\begin{array}{l} \mathcal{K}_x = \{x \in \mathbb{R}^{n_1} : Ax \geq a,\ x \geq 0\} \\ \mathcal{K}_y = \{y \in \mathbb{R}^{n_2} : By \geq b,\ y \geq 0\} \end{array} \tag{26}$$

where $a \in \mathbb{R}^{m_1}, b \in \mathbb{R}^{m_2}, A \in \mathbb{R}^{m_1 \times n_1}$ and $B \in \mathbb{R}^{m_2 \times n_2}$. By this reason this bilinear program is usually called Disjoint and differs from the so-called Jointly Bilinear Program, in which the x_i and y_i variables appear together in at least one constraint [3].

The BLP has been studied by many authors in the past several years [3, 18, 19, 21, 24, 27, 29, 38]. A number of important applications of this problem has appeared in the literature [27] and many algorithms have been designed for finding a stationary point or a global minimum of the BLP [18, 19, 21, 24, 29, 38]. Despite its simplicity, even the problem of finding a stationary point for a BLP is considered to be NP-hard [36].

In this section we discuss the solution of the BLP by exploiting its reduction to a nonconvex problem consisting of the minimization of a linear function on a PGLCP, that is denoted by MINPGLCP. In order to get this nonconvex problem, we first rewrite the BLP in the following equivalent form

$$\text{Minimize}_y \{d^T y + \min_x \{(c + Hy)^T x : x \in \mathcal{K}_x\} : y \in \mathcal{K}_y\}$$

By applying the duality theory of linear programming to the inner linear program in the variables x, it is not difficult to show [24] that if the BLP has a global minimum then such a point is also the global minimum of the following nonconvex program

$$\begin{array}{ll} \text{Minimize} & d^T y + a^T u \\ \text{subject to} & c - A^T u + Hy \geq 0 \\ & -a + Ax \geq 0 \\ & u^T (Ax - a) = 0 \\ & x^T (c - A^T u + Hy) = 0 \\ & -b + By \geq 0 \\ & x \geq 0,\ y \geq 0,\ u \geq 0 \end{array}$$

By introducing the slack variables for the inequality constraints, we can rewrite this MINPGLCP in the following form

$$\begin{array}{llll} \text{Minimize} & & & \begin{bmatrix} 0 \\ a \end{bmatrix}^T \begin{bmatrix} x \\ u \end{bmatrix} + d^T y \\[2ex] \text{subject to} & \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} & = & \begin{bmatrix} c \\ -a \end{bmatrix} + \begin{bmatrix} 0 & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} H \\ 0 \end{bmatrix} y \\[2ex] & v & = & b - By \\[2ex] & & & \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} \geq 0, \begin{bmatrix} x \\ u \end{bmatrix} \geq 0, y \geq 0, \quad v \geq 0 \\[2ex] & & & \begin{bmatrix} w^1 \\ w^2 \end{bmatrix}^T \begin{bmatrix} x \\ u \end{bmatrix} = 0 \end{array} \tag{27}$$

Hence the BLP reduces to the minimization of a linear function on a constraint set consisting of a PGLCP with a PSD matrix

$$M = \begin{bmatrix} 0 & -A^T \\ A & 0 \end{bmatrix}$$

for the complementary variables $w = [w^1 \ w^2]^T$ and $z = [x \ u]^T$.

The equivalence between a BLP and the MINPGLCP (27) has suggested a sequential procedure for finding a global minimum of the BLP by exploiting a finite number of solutions of the PGLCP in such a way that the value of the linear function always decreases. An algorithm based on this idea has been designed in [22] and has been applied to BLPs and some others nonconvex problems with some success [21, 22, 23]. A drawback of this approach is that, apart from the first, all the GLCPs are NP-hard and there exists no direct or iterative algorithm to process them efficiently. In fact, the authors have used an enumerative algorithm [1, 20] to compute solutions of all the GLCPs required by the sequential procedure [21, 22, 23].

A nonenumerative algorithm for finding the global minimum for the MINPGLCP (27) by exploiting solutions of the PGLCP has still to be designed. By theorem 1 such solutions can be found by computing stationary points of the nonlinear program discussed in the previous section. We recall that there exists quite efficient software for finding such stationary points, since the constraints are simple lower bounds on the variables [4, 5, 6, 7, 10, 14, 34].

As stated before, there are some other important optimization problems that are related with the BLP. The Concave Quadratic Program (CQP) is certainly one of the most revelant of these problems. We recall that a CQP is defined as follows

$$\begin{array}{ll} \text{Minimize} & 2c^T x + x^T H x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array} \tag{28}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$ and $-H$ is a PSD matrix of order $n \times n$. As is discussed in [28], by duplicating the number of variables, it is possible to reduce the CQP into the following BLP

$$\begin{array}{ll} \text{Minimize} & c^T x + c^T y + x^T H y \\ \text{subject to} & Ax \geq b \ , Ay \geq b \\ & x \geq 0 \ , y \geq 0 \end{array} \tag{29}$$

The Concave Quadratic Programming Problem has found many applications in different areas. Among them, the problem of finding a feasible solution of a Zero-One Integer Program should be mentioned. This problem is defined as follows, find x and y such that

$$\begin{array}{l} Ax + By \geq b \\ y \geq 0, \ \ x_i \in \{0,1\}, \ \ i = 1,\ldots,n \end{array} \tag{30}$$

where $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times l}$ are given vector and matrices. This problem is obviously equilavent to the following CQP

$$\begin{array}{ll} \text{Minimize} & x^T(e - x) \\ \text{subject to} & Ax + By \geq b \\ & 0 \leq x \leq e, \ \ y \geq 0 \end{array} \tag{31}$$

where e is a vector of ones of order n. The so-called knapsack problem is an important example of (30) and is defined by

$$\begin{array}{l} a^T x = b_0 \\ x_i \in \{0,1\}, \ \ i = 1,\ldots,n \end{array} \tag{32}$$

where $a \in \mathbb{R}^n$ and $b_0 \in \mathbb{R}^1$ are given. This problem can be cast in the form

$$\begin{array}{ll} \text{Minimize} & e^T x - x^T x \\ \text{subject to} & 0 \leq x \leq e \\ & a^T x \geq b_0 \\ & a^T x \leq b_0 \end{array} \tag{33}$$

As stated before, this CQP is equivalent to the following BLP

$$\begin{array}{ll} \text{Minimize} & e^T x + e^T y - x^T y \\ \text{subject to} & 0 \le x \le e \quad 0 \le y \le e \\ & a^T x \ge b_0 \quad a^T y \ge b_0 \\ & a^T x \le b_0 \quad a^T y \le b_0 \end{array} \tag{34}$$

Then this BLP can be reduced to a MINPGLCP of the form

$$\begin{array}{ll} \text{Minimize} & c^T z + d^T y \\ \text{subject to} & w = q + Mz + Ny \\ & v = p + Sy \\ & v, w, z, y \ge 0 \\ & z^T w = 0 \end{array} \tag{35}$$

where M is a PSD matrix.

We have then shown in this section that any bilinear program can be reduced into a MINPGLCP of the form (35), where M is a PSD matrix. Concave Quadratic Programs and the problem of finding a feasible zero-one integer solution also reduce to MINPGLCPs by exploiting their equivalences to a BLP. So all these problems can be solved by finding a finite number of solutions of the PGLCP, which can be done by computing stationary points of the nonlinear program (12). In section 5 we report some computational experience on solving CQPs and knapsack problems by using the merit functions mentioned before.

4 Application to the LCP

As stated before, the Linear Complementarity Problem (LCP) consists of finding vectors $z \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$ such that

$$w = q + Mz, \quad z \ge 0, \quad w \ge 0, \quad z^T w = 0 \tag{36}$$

where $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$ are given vector and matrix. It is known that the complexity of the solution of this problem is related with the class of the matrix M [8, 35]. If M is a RS matrix, there is a number of direct and iterative algorithms that process efficiently the LCP [8]. Furthermore the LCP can be solved in polynomial time if M is a PSD matrix [26, 35, 42]. In general the LCP is a NP-hard problem [18, 35] and only an enumerative algorithm is able to find a solution or to show that none exists [1, 20, 39].

The complexity of the LCP has motivated the search for techniques that exploit its reduction into a global optimization problem. The easiest of these forms is to put the nonlinear constraint $z^T w = 0$ in an objective function and get the following Joint Bilinear Program (JBLP)

$$\begin{array}{ll} \text{Minimize} & z^T w \\ \text{subject to} & w = q + Mz \\ & z \ge 0, \quad w \ge 0 \end{array} \tag{37}$$

It is then obvious that a LCP has a solution if and only if this JBLP has a global minimum with zero value. An enumerative algorithm has been proposed in [1, 20] for solving the LCP by finding a feasible solution of this JBLP with zero objective value. As is reported in [20], the incorporation of some efficient heuristics and a quadratic solver for finding stationary points of special quadratic programs has made this enumerative algorithm an interesting technique for processing difficult LCPs.

Recently a Disjoint Bilinear Programming formulation of the LCP has received some interest [2, 33]. This formulation is obtained in two stages. First, the LCP is rewritten as the following Augmented LCP

$$\begin{array}{l} \text{ALCP}: \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} e \\ q \end{bmatrix} + \begin{bmatrix} 0 & -I \\ M & 0 \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} \\ \qquad \begin{bmatrix} z \\ x \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \ge 0, \begin{bmatrix} z \\ x \end{bmatrix}^T \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0 \end{array} \tag{38}$$

where I is the identity matrix of order n and e is a vector of ones of order n. Now, moving the nonlinear constraint $z^T\alpha + x^T\beta = 0$ into an objective function, allows the ALCP to be written as the following Disjoint BLP

$$\begin{array}{ll} \text{Minimize} & e^Tz + q^Tx + x^T(M-I)z \\ \text{subject to} & Mz \geq -q \quad 0 \leq x \leq e \\ & \quad z \geq 0 \end{array}$$

Furthermore the LCP has a solution if and only if the optimal value of the BLP is zero.

As discussed in the previous section, we can transform this BLP into the following MINPGLCP

$$\begin{array}{llll} \text{Minimize} & & & e^Tz - e^Tu \\ \text{subject to} & \begin{bmatrix} w \\ \beta \end{bmatrix} & = & \begin{bmatrix} q \\ e \end{bmatrix} + \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} M-I \\ 0 \end{bmatrix} z \\ & \alpha & = & q + Mz \\ & & & \begin{bmatrix} x \\ u \end{bmatrix},\ z,\ \begin{bmatrix} w \\ \beta \end{bmatrix},\ \alpha \geq 0 \\ & & & \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} w \\ \beta \end{bmatrix} = 0 \end{array}$$

where the matrix corresponding to the complementary variables is PSD. So we can find a solution to the LCP by computing a solution of the PGLCP with $e^Tz - e^Tu = 0$. We can obviously add this constraint to the PGLCP but this destroys the structure of the PGLCP and the problem becomes NP-hard. A simple alternative way is to introduce the constraint

$$\gamma_0 = e^Tz - e^Tu$$

and a column with a parameter $\lambda_0 \geq 0$ that is complementary to γ_0. This leads into the following GLCP

$$\begin{array}{lll} \begin{bmatrix} w \\ \beta \\ \gamma_0 \end{bmatrix} & = & \begin{bmatrix} q \\ e \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & I & 0 \\ -I & 0 & -e \\ 0 & e^T & 0 \end{bmatrix} \begin{bmatrix} x \\ u \\ \lambda_0 \end{bmatrix} + \begin{bmatrix} M-I \\ 0 \\ -e^T \end{bmatrix} z \\ \alpha & = & q + Mz \\ & & \alpha, z, w, \beta, \gamma_0, x, u, \lambda_0 \geq 0 \\ & & x^Tw = u^T\beta = \lambda_0\gamma_0 = 0 \end{array} \tag{39}$$

It is easy to see that the matrix corresponding to the complementary variables

$$\begin{bmatrix} 0 & I & 0 \\ -I & 0 & -e \\ 0 & e^T & 0 \end{bmatrix} \tag{40}$$

is PSD. Hence the problem reduces to a PGLCP of the form discussed in section 2. It is now important to know when a solution of this PGLCP leads into a solution of the LCP. To do this, let

$$\mathcal{K} = \{z \in \mathbb{R}^n : q + Mz \geq 0,\ \ z \geq 0\}$$

be the feasible set of the LCP. Then the following result holds:

Theorem 2 *If $\mathcal{K} \neq \emptyset$ then the PGLCP has solution $(\bar{x}, \bar{z}, \bar{u}, \bar{\lambda}_0, \bar{w}, \bar{\beta}, \bar{\gamma}_0, \bar{\alpha})$ such that $\bar{\lambda}_0 \leq 1$. Furthermore $(\bar{z}, \bar{w})$ is a solution of the LCP provided $\bar{\lambda}_0 < 1$.*

Proof: Since $\mathcal{K} \neq \emptyset$, there exists at least a $\bar{z} \geq 0$ such that $q + M\bar{z} \geq 0$. Let

$$\begin{array}{l} \bar{w} = \bar{\alpha} = q + M\bar{z} \\ \bar{u} = \bar{z},\ \ \bar{\gamma}_0 = e^T\bar{u} - e^T\bar{z} = 0 \\ \bar{\lambda}_0 = 0,\ \ \bar{\beta} = e,\ \ \bar{x} = 0 \end{array}$$

Hence $(\bar{x}, \bar{z}, \bar{u}, \bar{\lambda}_0, \bar{w}, \bar{\beta}, \bar{\gamma}_0, \bar{\alpha})$ belongs to the set $\bar{\mathcal{K}}$ of the linear constraints of the PGLCP (39). Since the matrix (40) is PSD, any stationary point of

$$\begin{array}{ll} \text{Minimize} & w^T x + \beta^T u + \lambda_0 \gamma_0 \\ \text{subject to} & (x, z, u, \lambda_0, w, \beta, \gamma_0, \alpha) \in \bar{\mathcal{K}} \end{array}$$

is a solution of the PGLCP [25]. As the objective function of this program is bounded from below on $\bar{\mathcal{K}}$ and $\bar{\mathcal{K}} \neq \emptyset$ such a stationary point exists [4, 35]. Hence the PGLCP has at least a solution.

Now let $(\bar{x}, \bar{z}, \bar{u}, \bar{\lambda}_0, \bar{w}, \bar{\beta}, \bar{\gamma}_0, \bar{\alpha})$ be a solution of the PGLCP (39). It follows from the definition of this problem that

$$0 \leq \bar{\beta}_i + \bar{x}_i = 1 - \bar{\lambda}_0$$

Hence $\bar{\lambda}_0 \leq 1$ and this proves the first part of the theorem.

To prove the second part, consider a solution $(\bar{x}, \bar{z}, \bar{u}, \bar{\lambda}_0, \bar{w}, \bar{\beta}, \bar{\gamma}_0, \bar{\alpha})$ of the PGLCP (39) with $\bar{\lambda}_0 < 1$. Then there are two possible cases:

i) $\bar{x}_i > 0$. Hence $\bar{w}_i = 0$ and $(q + M\bar{z})_i + (\bar{u}_i - \bar{z}_i) = 0$. Then $\bar{\alpha}_i + \bar{u}_i - \bar{z}_i = 0$ which implies $\bar{u}_i \leq \bar{z}_i$.

ii) $\bar{x}_i = 0$. Then $\bar{\beta}_i = 1 - \bar{\lambda}_0 - \bar{x}_i = 1 - \bar{\lambda}_0 > 0$. Hence $\bar{u}_i = 0$ and $\bar{w}_i = (q + M\bar{z})_i - \bar{z}_i$. If $\bar{z}_i > 0$ then $e^T\bar{u} < e^T\bar{z}$ and $\bar{\gamma}_0 = e^T\bar{u} - e^T\bar{z} < 0$, which is impossible. So $\bar{z}_i = 0$.

We have then shown that

$$\begin{cases} \bar{u}_i \leq \bar{z}_i \ \text{ for all } \ i \ \text{ such that } \ \bar{x}_i > 0 \\ \bar{u}_i = \bar{z}_i = 0 \ \text{ for all } \ i \ \text{ such that } \ \bar{x}_i = 0 \end{cases}$$

Since $e^T\bar{u} \geq e^T\bar{z}$, then $\bar{u} = \bar{z}$ and

$$\bar{w} = q + \bar{u} + (M - I)\bar{z} = q + M\bar{z} = \bar{\alpha}$$

To prove that $\bar{z}$ is a solution of the LCP, it is sufficient to show that $\bar{z}_i > 0$ implies $\bar{w}_i = 0$. But if $\bar{z}_i > 0$ then $\bar{u}_i = \bar{z}_i > 0$ and

$$\bar{\beta}_i = 0 = 1 - \bar{\lambda}_0 - \bar{x}_i$$

Since $\bar{\lambda}_0 < 1$, then $\bar{x}_i > 0$ and $\bar{w}_i = 0$. This proves the theorem. □

This theorem enables us to solve the LCP by processing the PGLCP (39). Since the matrix (40) is PSD, a solution to this PGLCP can be found by computing a stationary point of the associated nonlinear program with zero lower bounds discussed in section 2. After finding such a point, there are two possible cases:

i) $\bar{\lambda}_0 < 1$ and $(\bar{z}, \bar{w})$ is a solution of the LCP.

ii) $\bar{\lambda}_0 = 1$ and $(\bar{z}, \bar{w})$ may be a solution of the LCP or not.

In the first case, the procedure has found a solution of the LCP. If in the second case $(\bar{z}, \bar{w})$ is a not solution of the LCP ($\bar{z}_i\bar{w}_i > 0$ for at least one i) then another stationary point of the associated nonlinear program has to be computed.

The computation of stationary points for the nonlinear program (12) associated to the PGLCP (39) can nowadays be done in a very efficient way, as there exist good algorithms to perform this task [4, 5, 6, 7, 10, 14, 34]. The main problem is to get an initial point for the algorithm that leads into a stationary point that is also a solution of the LCP. Recently there has been some work on the design of procedures that try to find a global minimum for an optimization problem by a clever choice of initial points. A heuristic procedure of this type has to be designed in order to fully exploit theorem 1 for the solution of LCPs.

5 Computational Experience

In this section we report some computational experience on a Digital Alpha Server 5/300, running Digital Unix 4.0B, with PGLCPs that arise on the solution of bilinear programs and LCPs by exploiting their reformulations discussed in the two previous sections. These PGLCPs are solved by computing stationary points of merit functions of the form (12). We have chosen the code LANCELOT [7] for such a task, since it is nowadays accepted as a robust code for processing these type of nonlinear programs.

In our first experience, we have solved some GLCPs that are the constraint sets of MINPGLCPs associated with Concave Quadratic Programs (CQP), according to the developments described in section 3. We start by describing the test problems $PQ10, \dots, PQ18$ that have been used in our experiences.

i) $PQ10$ – These are knapsack problems of the form

$$\begin{array}{ll} \text{Minimize} & e^T x - x^T x \\ \text{subject to} & a^T x \leq b \\ & a^T x \geq b \\ & 0 \leq x \leq e \end{array}$$

where $e \in \mathbb{R}^n$ is a vector of ones, $a \in \mathbb{R}^n$ is a vector whose components are random numbers belonging to the interval $[1, 50]$ and b is a positive real number satisfying

$$b = \sum_{i \in I} a_i$$

Here I is a subset of $\{1, \dots, n\}$ with cardinal $\frac{n}{4}$ $[PQ10(\frac{n}{4})]$, $\frac{n}{2}$ $[PQ10(\frac{n}{2})]$ and $\frac{3n}{4}$ $[PQ10(\frac{3n}{4})]$. These knapsack problems are transformed into MINPGLCPs according to the process explained in section 3. The constraint set of this nonconvex program was the PGLCP to be solved in this experience.

ii) $PQ11, \dots, PQ18$ – These are the CQP test problems described in [13]. As before these CQPs are transformed into MINPGLCPs and the PGLCPs to be tested in our experiences are the constraint sets of these nonconvex programs.

The results of the solution of the GLCPs by using the merit functions (23) and (24) are displayed in Table 1 under the headings $OPT1$ and $OPT2$ respectively. In this Table, IT and CPU represent the number of iterations and CPU time taken by LANCELOT to get a stationary point for these functions. Furthermore $VALUEF$ gives the value of the corresponding merit function at this stationary point. We recall that in theory this value should be equal to zero. By looking to the figures presented in Table 1, we come to the conclusion that LANCELOT is able to find a stationary point for both the merit functions in a reasonable amount of iterations and CPU time. In fact, only three problems require more than 30 iterations to get a stationary point for the merit function (23). Furthermore LANCELOT usually requires more iterations for finding a stationary point to the merit function (24), but the gap is not large. It is, however, interesting to note that the ratio $\frac{CPU\text{time}}{IT}$ is usually smaller for this latter function.

In our second experience we have considered some LCPs taken from known sources and their equivalent PGLCPs that are obtained according to the process explained in section 4. As before, these PGLCPs are solved by computing stationary points of the merit functions (23) and (24). We start by describing the test problems used in this second experience.

$PROB1$ – This is the LCP discussed in [35], where $q \in \mathbb{R}^n$ is a vector with all components equal to -1 and M is a lower triangular P–matrix defined by

$$\begin{array}{l} m_{ii} = 1, \ \ i = 1, \dots, n \\ m_{ij} = 2 \ \text{ for } \ i > j \\ m_{ij} = 0 \ \text{ for } \ i < j \end{array}$$

Table 1: Solution of PGLCPs associated with Concave Quadratic Programs.

		$OPT1$			$OPT2$		
	N	IT	CPU	$VALUEF$	IT	CPU	$VALUEF$
$PQ10(\frac{n}{4})$	20	6	1.33	1.54e-10	6	1.00	1.86e-14
	50	6	5.77	1.64e-10	11	7.58	2.57e-11
	100	7	25.33	2.47e-09	13	17.39	2.51e-11
	150	5	65.16	3.31e-11	14	36.45	4.01e-11
$PQ10(\frac{n}{2})$	20	6	1.43	1.89e-11	6	1.02	1.82e-11
	50	6	6.62	1.68e-10	7	4.21	1.97e-11
	100	6	27.15	2.48e-10	7	12.95	1.18e-10
	150	6	66.79	3.39e-10	7	32.02	4.17e-11
$PQ10(\frac{3n}{4})$	20	6	1.62	8.20e-12	6	1.06	1.07e-11
	50	6	7.76	1.46e-09	7	3.92	1.30e-10
	100	7	33.89	4.71e-10	7	10.88	7.86e-11
	150	7	89.20	1.90e-08	7	31.93	1.53e-10
$PQ11$		7	0.11	5.03e-15	7	0.11	2.99e-14
$PQ12$		7	0.13	1.77e-27	9	0.15	2.16e-11
$PQ13$		7	0.50	1.47e-10	13	0.63	8.70e-11
$PQ14$		10	0.18	2.17e-09	7	0.13	1.56e-13
$PQ15$		33	4.79	2.79e-02	34	3.68	2.79e-02
$PQ16$		10	0.77	1.05e-11	14	0.65	3.64e-03
$PQ17.1$		14	1.68	6.14e-11	22	2.01	6.75e-12
$PQ17.2$		17	1.97	2.63e-11	39	2.83	4.18e-12
$PQ17.3$		52	8.56	2.89e-14	53	4.05	1.81e-12
$PQ17.4$		19	1.36	4.83e-10	16	1.04	3.36e-13
$PQ17.5$		40	4.78	2.92e-12	53	4.19	4.09e-02
$PQ18$		20	3.10	9.65e-11	13	2.47	2.41e-09

$PROB2$ – This LCP also appears in [35] and is defined by $q_i = -1$ for all $i = 1, \ldots, n$ and $M = L^T L$, where L is the matrix of $PROB1$. Hence M is a symmetric positive definite matrix.

$PROB3$ – This LCP has been introduced by Chadrasekaran, Pang and Stone and is also presented in [35]. The vector q also satisfies $q_i = -1$ for all $i = 1, \ldots, n$ and the matrix M is defined as follows:

$$\begin{array}{l} m_{ii} = 1, \;\; i = 1, \ldots, n \\ m_{ij} = 2 \;\text{ if }\; j > i \;\text{ and }\; i+j \;\text{ is odd} \\ m_{ij} = -1 \;\text{ if }\; j > i \;\text{ and }\; i+j \;\text{ is even} \\ m_{ij} = -1 \;\text{ if }\; j < i \;\text{ and }\; i+j \;\text{ is odd} \\ m_{ij} = 2 \;\text{ if }\; j < i \;\text{ and }\; i+j \;\text{ is even} \end{array}$$

It is possible to show that M is a positive semi–definite matrix.

$PROB4$ – This problem is also discussed in [35] and considers the vector q such that $q_i = -1$ for all $i = 1, \ldots, n$ and M to be the well–known Hilbert matrix defined by

$$m_{ij} = \frac{1}{i+j-1}$$

for all $i, j = 1, \ldots, n$. Again M is a positive semi-definite matrix.

$PROB5, 6, 7$ – Consider again the knapsack problem

$$a^T z = b, \;\; z_i \in \{0, 1\}, \;\; i = 1, \ldots, n$$

where, as before, the components of the vector a are random numbers belonging to the interval $[1, 50]$ and b is the positive real number

$$b = \sum_{i=1}^{\frac{n}{2}} a_i$$

The LCPs of $PROB5, 6$ and 7 are LCP formulations of this problem that appeared in the literature [26, 35, 36]. To get $PROB5$, we consider the LCP defined by

$$q = \begin{bmatrix} e \\ -b \\ b \end{bmatrix} \;\; , \;\; M = \begin{bmatrix} -I_n & 0 & 0 \\ a^T & -\alpha & 0 \\ -a^T & 0 & -\beta \end{bmatrix}$$

where I_n is the identity matrix of order n and $e \in \mathbb{R}^n$ is a vector of ones. The constants α and β are chosen in order M to be negative semi–definite (NSD) or indefinite (IND). For the first case α and β should satisfy

$$\alpha > k\frac{a^T a}{4} \;\; , \;\; \beta > k\frac{\alpha a^T a}{4\alpha - a^T a}$$

where k is a real number greater than one. This leads into $PROB5$(NSD). On the other hand, M is IND if α and β satisfy

$$\alpha > k\frac{a^T a}{4} \;\; , \;\; \beta < \frac{1}{k}\frac{\alpha a^T a}{4\alpha - a^T a}$$

These choices of α and β lead into $PROB5$(IND).

$PROB6$ is the LCP formulation of the knapsack problem discussed in [36] and is given by

$$q = \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ -b \\ b \end{bmatrix} \;\; , \;\; M = \begin{bmatrix} -I_n & e & -e \\ e^T & -2n & 0 \\ -e^T & 0 & -2n \end{bmatrix}$$

where, as before, e is a vector of ones and I_n is the identity matrix of order n. As is discussed in [36] the matrix M is symmetric negative semi–definite.

Finally $PROB7$ is the LCP formulation discussed in [26] and considers

$$q = \begin{bmatrix} p \\ p \\ \vdots \\ p \\ -b \\ b \end{bmatrix} \quad , \quad M = \begin{bmatrix} B & 0 & \cdots & 0 & 0 & 0 \\ 0 & B & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & B & 0 & 0 \\ & & \bar{a}^T & & & \\ & & -\bar{a}^T & & & \end{bmatrix}$$

where

$$p = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad , \quad \bar{a} = (\bar{a}_1, \ldots, \bar{a}_{4n+2})^T \in \mathbb{R}^{4n+2}$$

with

$$\bar{a}_i = \begin{cases} a_i & \text{if } i = 4j-3 \ , \quad j = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

As dicussed in [26] the matrix M has nonnegative principal minors (that is, $M \in P_0$), but it is not positive semi–definite.

$PROB8, 9$ – These are structured LCPs that are formulations of nonzero–sum bimatrix games [8, 35]. The LCP of $PROB8$ takes the form

$$q = \begin{bmatrix} -e^m \\ -e^r \end{bmatrix} \quad , \quad M = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$$

where e^j is a vector of ones of order j and A, B are positive matrices whose elements are random numbers belonging to the interval $[1, 50]$. On the other hand the vector q and the matrix M of the LCP corresponding to $PROB9$ are given by

$$q = \begin{bmatrix} -e^m \\ e^r \end{bmatrix} \quad , \quad M = \begin{bmatrix} 0 & A \\ -B & 0 \end{bmatrix}$$

For each one of these LCPs, we have generated four problems differing on its dimension n.

The results of the solution of these LCP test problems by processing their equivalent PGLCPs are displayed in Table 2. We recall that the LCP is equivalent to a PGLCP with a parameter λ_0. If $\lambda_0 < 1$ in a solution of this PGLCP, then a solution of the LCP is at hand. Otherwise ($\lambda_0 = 1$) no conclusion can be drawn about the existence of a solution to the LCP, but usually the solution of the PGLCP does not lead to a solution of the LCP. As before, the PGLCP is solved by computing the stationary point of the associated merit functions (23) or (24). The computational effort for performing such a task is displayed under the headings $OPT1$ and $OPT2$ respectively, by stating the number of iterations (IT) and the CPU time (CPU) that LANCELOT has required to get a stationary point in each one of the cases. In this Table, $VALUEF$ continues to represent the value of the merit function at this stationary point and $LAMBDA$ gives the value of the variable λ_0 at this solution. So $LAMBDA < 1$ means that a solution of the LCP has been found. Furthermore we write an asterisk when a solution of the LCP has been found in the case of $LAMBDA = 1$.

The results displayed in Table 2 lead to conclusions similar to the case of CQPs about the ability of LANCELOT to get stationary points that are solutions of the PGLCP in a small amount of effort. As before, the natural function (23) seems to be a better choice in terms of the number of iterations that LANCELOT requires to get a stationary point that solves the PGLCP. The results

Table 2: Solution of PGLCPs associated with LCPs.

		$OPT1$				$OPT2$			
	N	IT	CPU	$VALUEF$	$LAMBDA$	IT	CPU	$VALUEF$	$LAMBDA$
$PROB1$	20	6	0.08	1.28e-21	5.01e-01	9	0.16	1.84e-15	4.18e-01
	50	6	0.36	5.44e-19	5.83e-01	10	0.72	3.91e-20	1.00e+00
	100	8	2.27	5.48e-14	5.12e-01	9	2.18	2.15e-27	1.00e+00*
	150	9	5.59	2.06e-25	5.87e-01	10	4.82	6.78e-14	6.83e-01
$PROB2$	20	7	0.11	2.19e-20	0.00e+00	9	0.15	2.70e-27	0.00e+00
	50	10	0.71	2.07e-14	1.00e+00*	5	0.47	1.16e-28	1.00e+00*
	100	7	1.98	2.88e-28	1.00e+00*	6	1.67	7.01e-28	1.00e+00*
	150	9	4.83	7.64e-15	1.00e+00	7	5.05	5.44e-27	1.00e+00*
$PROB3$	20	8	0.45	2.46e-09	0.00e+00	11	0.55	7.72e-12	1.00e+00
	50	7	5.70	3.39e-09	2.43e-03	17	8.27	2.85e-12	1.00e+00
	100	7	29.93	7.52e-09	4.90e-01	21	71.98	8.82e-10	6.65e-01
	150	8	85.28	3.30e-08	5.05e-01	20	176.98	1.82e-10	5.18e-01
$PROB4$	20	10	0.24	1.25e-10	1.00e+00	14	0.40	9.20e-13	1.00e+00
	50	11	1.23	1.80e-11	1.00e+00	12	1.87	5.07e-14	1.00e+00
	100	12	7.65	5.02e-12	1.00e+00	16	14.53	1.17e-14	1.00e+00
	150	12	24.41	2.15e-10	1.00e+00	21	65.04	2.59e-15	1.00e+00
$PROB5$	20	12	0.16	1.34e-12	1.00e+00	8	0.23	2.09e-13	1.00e+00
(NSD)	50	8	0.57	1.29e-07	1.00e+00	14	1.10	1.00e-14	1.00e+00
	100	12	4.32	2.91e-11	1.00e+00	8	2.95	2.14e-12	1.00e+00
	150	10	3.08	1.02e-11	1.00e+00	10	7.52	3.12e-15	1.00e+00
$PROB5$	20	12	0.16	1.34e-12	1.00e+00	8	0.23	2.09e-13	1.00e+00
(IND)	50	8	0.57	1.29e-07	1.00e+00	14	1.10	1.00e-14	1.00e+00
	100	12	4.27	2.91e-11	1.00e+00	8	2.94	2.14e-12	1.00e+00
	150	10	3.06	1.02e-11	1.00e+00	10	7.49	3.12e-15	1.00e+00
$PROB6$	20	13	0.18	1.74e-12	1.00e+00	11	0.33	2.77e-14	1.00e+00
	50	13	0.75	7.97e-13	1.00e+00	12	1.29	1.30e-13	1.00e+00
	100	14	1.77	9.96e-11	1.00e+00	32	7.51	1.89e-12	1.00e+00
	150	14	4.37	9.36e-11	1.00e+00	34	10.70	6.68e-16	1.00e+00
$PROB7$	20	7	0.18	6.75e-11	1.00e+00	14	0.49	1.88e-14	1.00e+00
	50	8	0.43	1.53e-11	1.00e+00	8	0.66	2.31e-14	1.00e+00
	100	7	1.67	1.02e-10	1.00e+00	8	1.70	1.56e-14	1.00e+00
	150	8	4.72	2.46e-11	1.00e+00	11	12.66	5.72e-13	1.00e+00
$PROB8$	20	17	3.06	1.13e-07	8.35e-01	13	0.46	2.14e-13	1.00e+00
	50	9	8.97	3.45e-15	1.00e+00*	22	11.60	2.67e-14	1.00e+00*
	100	20	198.73	6.88e-08	8.06e-01	27	75.29	1.36e-06	5.23e-01
	150	14	397.49	3.61e-08	8.56e-01	30	646.96	1.61e-06	5.96e-01
$PROB9$	20	10	1.40	8.69e-15	1.00e+00*	12	0.76	2.06e-14	1.00e+00*
	50	17	27.92	1.32e-08	9.56e-01	18	22.77	5.17e-14	1.00e+00*
	100	15	171.00	5.09e-09	9.64e-01	27	103.27	1.57e-06	5.42e-01
	150	13	282.05	5.05e-08	8.58e-01	18	464.32	2.55e-14	1.00e+00*

also show that in general the solution of the PGLCP is not a solution of the LCP. However, we have only presented the numerical results that have been achieved by using LANCELOT with its recommended starting point. A heuristic procedure to get a good initial point for the local solver (LANCELOT or other) that leads into a stationary point of the merit function that is also a solution of the LCP will certainly be an important topic for future research. This will enable solving NP–hard LCPs and bilinear programs by nonenumerative techniques.

6 Conclusions

In this paper we have showed that a solution of a polynomial General Linear Complementarity Problem (PGLCP) can be found by computing a stationary point of an appropriate merit function. This result has important implications on the solution of bilinear and concave quadratic programs and zero–one integer programming problems. We have also showed that any Linear Complementarity Problem (LCP) can be reduced into a PGLCP. Hence, under certain conditions, a solution of the LCP can be found by computing a stationary point of an appropriate merit function.

Some computational experience with concave quadratic programs, knapsack problems and LCPs was included and showed the appropriateness of solving the associated PGLCPs by computing stationary points of the corresponding merit functions. We believe that these conclusions will have an important effect on the solution of these difficult nonconvex problems, particularly if we can design heuristic procedures capable of providing good starting points for the local search techniques that are employed to solve the PGLCP. This is a topic that deserves research in the future.

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