An integer programming model for truss topology optimization

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Abstract

In this paper a truss-structure model is described for finding a kinematically stable structure with optimal topology and cross-sectional size and minimum volume. The underlying model finds applications in some civil engineering structural design problems and takes into consideration all the conditions associated with the limit states usually presented in structural safety codes. Ultimate limit states are treated applying plasticity theory, while serviceability limit states are dealt with via elasticity theory. The admissible solution space is discretised using bar elements. A $0-1$ variable is assigned to each one of these elements, in order to indicate if it is or not included in the solution. The mathematical formulation of the model leads to a mixed $0-1$ integer nonlinear program with a nonlinear objective function and linear and bilinear constraints. It is shown that this problem can be reduced into a mixed $0-1$ integer linear program by exploiting the so–called reformulation–linearization technique. Some computational experience is included to highlight the importance of these formulations in practice.

Key Words: Truss topology optimization, integer programming, reformulation-linearization technique.

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1 Introduction

A decision support system to get the optimal skeleton form of the structure that supports the applied loads can become a quite useful tool for an engineer in the design of a structure. This has motivated an area of intense research [6, 15, 16, 17, 18, 20, 25, 26, 28, 5]. Most of these papers describe trusses that, due to its simplicity, are ideal for the study of properties and characteristics associated to optimal structures. The results of this research have found interest in coverings, bridges and towers high-voltage.

Some optimization approaches have been developed for finding optimal trusses. The most challenged one is Topological Optimization, where some elements of an initial structure may be removed in order to get an optimal subset of elements. The structural model changes during the process and this turns out to be one of the most interesting problems in structural optimization. Another important characteristic of these models for civil engineering structures is the fact that the matrix associated with the connection of the bars in the resultant optimal substructure must have full row rank. This condition can not be analytically represented as a constraint, turning the optimization problem even more difficult to process.

Some research has been developed in the last four decades in this area, the majority related with sizing and geometrical optimization of optimal structural design. Due to its complexity, few contributions have been done in optimal topology. However, its importance in structural optimization is recognized since it allows a substantial gain of material and a significant improvement in the design of the structure.

The topological optimization of discrete structures was introduced by Dorn et al. (1964) [9], who applied a linear programming method to optimize truss topology. Since then much research has been developed in this area [19, 28]. In most of these studies, the cross-sectional area of each bar may take a zero value, in which case the bar is eliminated from the structure. Moreover, these variables assume real values in order to make easier the solution process. As is shown in [8, 24] some of these bars cannot be eliminated from the structure, whence the structural problem should have a combinatorial nature.

Another drawback of these studies is the possibility of the optimal topology to correspond to a singular point of the design admissible domain. This fact is a consequence of the discontinuity of some constraints when the cross-sectional area is zero. The singularity of the optimal topology in trusses was firstly shown in [24] and since then it has been a subject of intense research [14, 15, 16, 17, 18]. The substitution of a discrete variable $x_i \in \{0, 1\}$ by a continuous ones $0 \leq x_i \leq 1$ is much too strong. Thus for these models to be useful in practice, the variables associated to the cross-sectional area of each bar must be considered as discrete. In this case the difficulty associated with the discontinuity of some constraints completely disappear. Some algorithms have been developed for processing this type of problems [3, 4, 25, 26]. More recently, Bollapragada et al. [6] presented a topological and sizing optimization model that has been formulated as a mixed integer linear program [11].

Another aspect that distinguishes the different contributions in this area is the manner how the limit states for the structural safety codes are defined. In civil engineering structural problems, both ultimate and serviceability limit states have to be considered. The former corresponds to collapse or other forms of structural ruin, while the latter are related to scenarios that should be only reached in extreme circumstances. Such scenarios involve excessive displacements and deformations, vibrations that may cause discomfort or alarm, and damage affecting the form, durability or the use of the structure.

In the ultimate limit states the effect of the design loads, forces or stresses should not exceed the design values of the structural resistance, admissible forces or stresses. On the other hand in the serviceability limit states the effects should comply with criteria of good performance, that give limits to the displacements to their maximum admissible values.

Structural optimization usually reports to plastic models or to elastic models. In the former case, the ultimate limit states are treated using the Plasticity Theory and serviceability limit states are not considered [22, 27]. The elastic models deal with both ultimate and serviceability limit states by means of the Elasticity Theory [29, 6].

In this paper a mixed model has been adopted using Elasticity Theory for serviceability limit states and Plasticity Theory for ultimate limit states. This model, here designed by elastoplastic, is in our opinion the best suited to the current trend of the safety codes [1]. As in [14, 15, 16, 17, 19, 24], the analysis presented in this paper is based on the trusses. Since the application of kinematically unstable trusses is confined to particular structures and special forces, one of our most important objectives is to eliminate from the feasible set the solutions associated with kinematically unstable trusses. So, each admissible solution is characterized by a vector x, whose components are assigned to the value 1 or 0, depending on the corresponding bar to be or not to be included in the feasible solution under consideration. The mathematical formulation leads to a mixed integer $0-1$ nonlinear program. By exploiting the so–called reformulation–linearization technique [21] it is possible to reduce this program into a mixed integer $0-1$ linear programming problem. Computational experience with the solution of some instances of this integer program shows that the formulation is a quite interesting tool for the design of a truss structure.

This paper is organized as follows. In section 2 the topological optimization model is introduced. Conditions for a truss to be kinematically stable are discussed in Section 3. A mixed–integer linear programming formulation of the model is fully presented in Section 4. The solution of some illustrative examples and some conclusions concerning the validity of the formulation are reported in the last section of the paper.

2 A topological optimization model

The admissible structural domain is referenced by a bidimensional cartesian system Oxy, in which the various alternative solutions for the problem under consideration can be developed. A discretisation [30] of this domain is then considered in which the mesh is composed by bar elements joined at the nodal points.

The structural domain is submitted to the various actions defined in the safety code [1] such as the structural self-weigth, wind, earthquake and so on. These actions lead to different l loading conditions, each of them is represented by nodal point loads

$$f^l = \begin{bmatrix} f_x^l \\ f_y^l \end{bmatrix}.$$

Some of these loads are reactions r^l, when the associated nodes are connected to the exterior. The nodal displacements

$$u^l = \begin{bmatrix} u_x^l \\ u_y^l \end{bmatrix}$$

are associated to these nodal forces. The stress field within each bar element i for loading condition l can be determined from its axial load e_i^l, while the strain field is given by the axial deformation d_i^l.

The fundamental conditions to be satisfied in the serviceability limit states are equilibrium, compatibility, boundary conditions and elastic constitutive relations of the structural material. As in structural civil engineering problems the displacements are generally accepted to be small, the fundamental conditions can be performed on the initial structures.

Equilibrium has to be verified at a nodal level and relates the elastic axial bar forces e_e^l with support reactions r_e^l and applied nodal loads f^l by

$$C^T e_e^l - B r_e^l - f^l = 0, \tag{1}$$

where C and B are matrices depending on the structural topology.

The compatibility conditions imply equal displacement for all the bar ends joining at the same node and can be expressed as

$$d_e^l = C u^l, \tag{2}$$

where d_e^l is the bar deformation vector, u^l is the nodal displacement vector and C is the connectivity matrix already used in (1).

The forces e_e^l in the structural bars are related to the bar deformations d_e^l by linear elastic constitutive relations given by the so-called Hooke's Law

$$e_e^l = K D_A d_e^l, \tag{3}$$

where $D_A = diag\{A_i\}$, with A_i a discrete variable associated to the cross-sectional area of bar i and $K = diag\{E_i h_i^{-1}\}$, with $E_i > 0$ the Young's modulus of bar i and h_i its length . It follows from (1), (2) and (3) that

$$C^T K D_A C u^l - B r_e^l - f^l = 0. \tag{4}$$

The structural boundary conditions are given by

$$u_m^l = 0 \tag{5}$$

for the nodes m connected to supports with zero displacement.

The nodal displacements should comply with the upper and lower bounds defined in the safety codes

$$u_{min} \leq u^l \leq u_{max}. \tag{6}$$

In structural civil engineering problems, ultimate limit states can be considered on the basis of the Plasticity Analysis. According to the Static Theorem of the Plasticity Theory, the fundamental conditions to be fulfilled by the structure are equilibrium, plasticity conditions and boundary conditions.

The equilibrium conditions are given in a similar form to (1) by

$$C^T e_p^l - B r_p^l - \lambda f^l = 0, \tag{7}$$

where e_p^l is the plastic force vector, r_p^l the plastic reaction vector and λ is a partial safety majoration factor for the nodal forces corresponding to the applied actions, prescribed in structural safety codes [1, 2].

The plasticity conditions can be expressed as

$$e_{min} \leq e_p^l \leq e_{max}, \tag{8}$$

where e_{min} and e_{max} are the minimum and maximum admissible values for the element forces defined in the code [2].

The conditions (4), (5), (6), (7) and (8) considered so far are satisfied by many solutions in which some bars have zero force. A vector x is further introduced in the model such that each variable x_i is associated with bar i and takes value 1 or 0, depending on the bar i to be or not to be included in the solution.

The force in a generic bar i can then be replaced by the product $x_i e_{p_i}^l$ yielding a null force in non-existing bars. So the axial bar force must verify the following conditions

$$D_x e_{min} \leq e_p^l \leq D_x e_{max}, \tag{9}$$

where

$$D_x = diag(x_i). \tag{10}$$

Furthermore the diagonal matrix D_A takes the form $D_A D_x$. The model seeks an optimal solution corresponding to the minimum use of structural material V. If A_i is the cross-sectional area of bar i and h_i is its length, then the objective function takes the form

$$V = \sum_i x_i A_i h_i. \tag{11}$$

3 Kinematic Stability

It follows from the description of the constrains and objective function presented in the previous section that the model can generate trusses which are not kinematically stable. In structural civil engineering problems the trusses must be kinematically stable that is, a mechanism can not be generate independently of the loads applied set.

A simple criterion to check a mechanism is the use of the Grubler's Criterion [12]. Let

$$DOF = 2nn^* - nb^* - m^* \tag{12}$$

be the degree of freedom of the truss with nn^* nodes, nb^* bars and m^* simple supports. If the corresponding truss is not a mechanism then $DOF \leq 0$.

In order to incorporate this criterion in the model, another vector $z \in \mathbb{R}^{nn}$ has to be introduced such that each variable z_n is associated with node $n \in \{1, \ldots, nn\}$ belonging to the initial mesh and takes a value 1 or 0 if it is or not included in the solution, that is, if there are or not bars connected with this node.

So the existence of such a node $n \in \{1, \ldots, nn\}$ can be stated as

$$z_n \leq \sum_{i \in I(n)} x_i \leq |I(n)| \, z_n, \tag{13}$$

where $I(n) \subset \{1, \ldots, nb\}$ is the set of bar indices i occurring at node n and its cardinal $|I(n)|$ is always different from zero. Note that nn and nb are the number of nodes and the number of bars of the initial mesh. So, if in the solution no bars occur in node n, $\sum_{i\in I(n)} x_i = 0$ and by the first inequality of (13) $z_n = 0$. Otherwise, if $\sum_{i\in I(n)} x_i \geq 1$ then by the second inequality of (13) $z_n = 1$.

Therefore, the Grubler's Criterion can be stated as

$$2 * \sum_{n=1}^{nn} z_n - \sum_{i=1}^{nb} x_i - \sum_{n=1}^{nn} s_n z_n \leq 0, \tag{14}$$

where s_n is the number of simple supports associated with node n.

This criterion eliminates many solutions associated to mechanisms. However, it does not give a sufficient condition for a structure to be kinematically stable. A further condition is required, that is associated with the rank of the matrix C^* of order $2nn^* \times (nb^* + na^*)$ matrix (with $nb^* = (\sum_{i=1}^{nb} x_i)$, $nn^* = \sum_{n=1}^{nn} z_n$ and $na^* = \sum_{n=1}^{nn} s_n z_n$), obtained by elimination of the rows and columns associated to $z_n = 0$ and the zero columns from the matrix $[-C^T D_x \quad B]$. Then [10] the structure is kinematically stable if C^* has full row rank. In order this condition to be satisfied by the optimal structure, the two following constraints similar to (7) and (8) are incorporated in the model

$$-C^T(D_x e_a) + B r_a + Z f_a = 0 \tag{15}$$

$$D_x e_{min} \leq D_x e_a \leq D_x e_{max}, \tag{16}$$

where Z is a $2nn \times 2nn$ diagonal matrix, with diagonal elements z_{jj} equal to z_n of the node n associated to the direction j and f_a is a vector of the perturbed nodal load applied in all directions.

Let f_a^* be the vector obtained by elimination of the rows associated to $z_n = 0$ from vector Zf_a. If the rank of matrix C^* is smaller than $2nn^*$, the rank of the augmented matrix $[C^* \ f_a^*]$ is $rank(C^*)+1$, since f_a^* is randomly generated and the constraints (15) are unfeasible. Thus the probability of the matrix C^* to have linearly dependent rows is null and matrix C^* should have full row rank in the set of feasible solutions.

To show the need for incorporating constraints (15) and (16) in the model, let us consider the following truss with only one nodal load applied on node II in direction Ox.

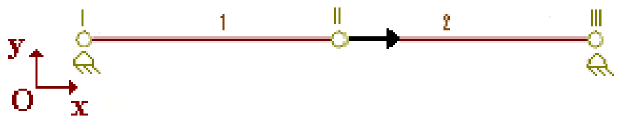

Figure 1: Truss kinematically unstable

The truss, shown in Figure 1, is not a mechanism, since $DOF = 0$. If the constraints (15) and (16) are not considered in the formulation of the model, this truss can be an optimal solution for the problem. However, the matriz C^* associated with this truss

$$C^* = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

has not full row rank and this truss is kinematically unstable.

Since, in constraint (15), $f_a \neq 0$ is a vector of perturbed nodal lodal applied in all direction and C^* has not full row rank, the constraints (15) are unfeasible. Thus kinematically unstable trusses are not feasible solutions for our model.

The model underlying to this problem can be formulated as the following mixed–integer $0-1$ nonlinear program:

$$\text{(P1)} \qquad \mathit{Minimize} \qquad V = \sum_{i=1}^{nb} x_i A_i h_i$$

subject to

$$M(D_A D_x)d^l - Br_e^l - f^l = 0 \tag{17}$$
$$d^l = Cu^l \tag{18}$$
$$u_{min_j} \leq u_j^l \leq u_{max_j} \tag{19}$$
$$u_{j_m}^l = 0 \tag{20}$$
$$-C^T e_p^l + Br_p^l + \lambda f^l = 0 \tag{21}$$
$$D_x D_A t_{min} \leq e_p^l \leq D_x D_A t_{max} \tag{22}$$
$$z_n \leq \sum_{i \in I(n)} x_i \leq |I(n)|\, z_n \tag{23}$$
$$2 * \sum_{n=1}^{nn} z_n - \sum_{i=1}^{nb} x_i - \sum_{n=1}^{nn} s_n z_n \leq 0 \tag{24}$$
$$-C^T e_a + Br_a + Zf_a = 0 \tag{25}$$
$$D_x D_A t_{min} \leq e_a \leq D_x D_A t_{max} \tag{26}$$
$$A_i \in S_i = \{A_{i1}, \ldots, A_{iN_i}\}, \tag{27}$$
$$x_i \in \{0, 1\},\ i = 1, \ldots, nb, \tag{28}$$

with $l = \{1, \ldots, nc\}$, $j = \{1, \ldots, 2nn\}$, $j_m = \{1, \ldots, na\}$, $n = \{1, \ldots, nn\}$ and $i = \{1, \ldots, nb\}$.

The meanings of the parameters in this program are presented below:

nb	number of bars;
na	number of simple supports;
nn	number of nodes;
nc	number of loading conditions;
N_i	number of discrete sizes available for cross-sectional area of bar i;
A_{ik}	k-th discrete size for bar i;
S_i	set of possible discrete cross-sectional area available for bar i;
C	$nb \times 2nn$ matrix of direction cosines relating bar forces with nodal directions;
B	$2nn \times na$ matrix of direction cosines relating nodal directions with nodal suports directions;
D_x	diagonal matrix, $[diag(x_i)]$;
D_A	diagonal matrix, $[diag(A_i)]$;

M	matrix $\left[C^T diag\left(\frac{E_i}{h_i}\right)\right]$;
E_i	Young's modulus of bar i;
h_i	length of bar i;
f_j^l	applied nodal loads in direction j for loading condition l;
$I(n)$	set of bars indices which occur in node n;
λ	safety factor;
$\|I(n)\|$	cardinal of set $I(n)$;
s_n	number of simple supports associated with node n;
Z	$2nn \times 2nn$ diagonal matrix, with z_{jj} equal to z_n of the node n associated to the direction j;
f_a	perturbed nodal load applied in all directions;
t_{min_i}	minimum stress in compression of bar i, where $t_{min_i} = \frac{e_{min_i}}{A_i}$;
t_{max_i}	maximum stress in tension of bar i, where $t_{max_i} = \frac{e_{max_i}}{A_i}$;
u_{min_j}, u_{max_j}	minimum and maximum nodal displacement in direction j.

The variables have the following meanings:

A_i	cross-sectional area of bar i;
x_i	$0-1$ variable stating whether the bar i exists or not;
$e_{p_i}^l$	bar force of bar i for loading condition l;
$r_{p_m}^l$	plastic reaction in supports m for loading condition l;
$r_{e_m}^l$	elastic reaction in supports m for loading condition l;
d_i^l	deformation of bar i for loading condition l;
u_j^l	nodal displacement in the direction j for loading condition l;
z_n	$0-1$ variable stating whether the node n exists or not;
e_{a_i}	bar force of bar i for the perturbed nodal load;
r_{a_m}	plastic reaction in supports m for the perturbed nodal load.

This mixed–integer $0-1$ nonlinear problem has $nc \times (4nn + 3nb) + 4nn + 2nb + 1$ constraints and $nc \times (2nb + 2nn + 2na) + 3nb + nn + na$ variables. A similar formulation has been used in [6, 13]. However, as discussed in [23], it represents a sizing problem rather than a topological optimization problem.

4 A mixed–integer linear programming formulation

The formulation (P1) described in Section 3 contains a bilinear objective function on the variables x_i and A_i and some linear and bilinear constraints. The existence of these bilinear functions imposes some limitations to the use of commercial software for processing mixed-integer programs. In this section formulation (P1) is reduced to a mixed-integer $0-1$ linear program by using the so–called reformulation–linearization technique (RLT) [21].

The RLT consists of two steps, namely the reformulation and the linearization. The reformulation phase defines a set of nonnegative variable factors, based on the various bound restrictions, and then forms products of these factors with the original constraints to generate another implied nonlinear constraints. In the linearization phase, an appropriate technique of substitution of variables is used to linearize these nonlinear constraints. In general, the original nonlinear program and the resulting linear program do not have the same optimal solutions. However, there are certain special cases in which those problems possess exactly the same optimal solution. In this section the mixed nonlinear $0-1$ integer program (P1) is shown to be equivalent to a mixed-integer $0-1$ linear program. To do this, the vectors d^l are assumed to be bounded, that is, there exist fixed constant vectors $d_{\min}$ and $d_{\max}$ such that

$$d_{min} \leq d^l \leq d_{max} \qquad l = 1, \ldots, nc. \tag{29}$$

The existence of theses bounds is a consequence of the conditions (18), (19) and (20). In the Reformulation Phase the constraints (29) are multiplied by x_i and $1 - x_i$, i=1,...,nb and the further constraints

$$x_i d_{min_i} \leq x_i d_i^l \leq x_i d_{max_i}$$
$$d_{min_i}(1 - x_i) \leq d_i^l - x_i d_i^l \leq d_{max_i}(1 - x_i)$$

are incorporated in (P1). In the Linearization Phase new variables v_i^l defined by

$$v_i^l = x_i d_i^l \tag{30}$$

are introduced. Using these two phases of RLT technique, the problem (P1) can be stated as follows:

$$\text{(P2)} \qquad Minimize \qquad V = \sum_{i=1}^{nb} x_i A_i h_i$$

$$subject\ \ to$$

$$M(D_A)v^l - Br_e^l - f^l = 0 \tag{31}$$
$$d^l = Cu^l \tag{32}$$
$$u_{j_m}^l = 0 \tag{33}$$
$$u_{min_j} \leq u_j^l \leq u_{max_j} \tag{34}$$
$$d_{min_i} x_i \leq v_i^l \leq d_{max_i} x_i \tag{35}$$
$$d_{min_i}(1 - x_i) \leq d_i^l - v_i^l \leq d_{max_i}(1 - x_i) \tag{36}$$
$$-C^T e_p^l + Br_p^l + \lambda f^l = 0 \tag{37}$$
$$D_x D_A t_{min} \leq e_p^l \leq D_x D_A t_{max} \tag{38}$$
$$z_n \leq \sum_{i \in I(n)} x_i \leq |I(n)|\, z_n \tag{39}$$
$$2 * \sum_{n=1}^{nn} z_n - \sum_{i=1}^{nb} x_i - \sum_{n=1}^{nn} s_n z_n \leq 0 \tag{40}$$
$$-C^T e_a + Br_a + Zf_a = 0 \tag{41}$$
$$D_x D_A t_{min} \leq e_a \leq D_x D_A t_{max} \tag{42}$$
$$A_i \in S_i = \{A_{i1}, \ldots, A_{iN_i}\} \tag{43}$$
$$x_i \in \{0, 1\}, \tag{44}$$

where $l = 1, \ldots, nc,\ j = 1, \ldots, 2nn,\ j_m = 1, \ldots, na,\ n = 1, \ldots, nn$ and $i = 1, \ldots, nb$.

Next, the equivalence between the programs (P1) and (P2) is established. Since the objective functions of the two problems are the same, then it is enough to prove that there is a bijective correspondence between the feasible solutions of the two problems. If the vector $\bar{v}^l$, of components $\bar{v}_i^l = \bar{x}_i \bar{d}_i^l$, is introduced then by definition of the variables x_i it is easy to conclude that $(\bar{x}, \bar{a}, \bar{e}_p^l, \bar{r}_p^l, \bar{d}^l, \bar{u}^l, \bar{r}_e^l)$ is a feasible solution of (P1) if and only if $(\bar{x}, \bar{a}, \bar{e}_p^l, \bar{r}_p^l, \bar{v}^l, \bar{d}^l, \bar{u}^l, \bar{r}_e^l)$ is feasible for (P2). So the two formulations are equivalent.

In problem (P2), for each bar i, the cross-sectional area A_i takes a discrete value in the set S_i. According to the cardinal of these sets, two cases may occur and are discussed below.

- If $\#S_i = 1$ for all i, then (P2) is a mixed–integer linear program (MILP).
- If there exists an i such that $\#S_i > 1$, then (P2) is a mixed–integer nonlinear program (MINLP).

In the latter case, problem (P2) can be also reformulated as a MILP. For that purpose, let

$$A_i x_i = \sum_{k=1}^{N_i} A_{ik} y_{ik}, \tag{45}$$

where $A_i \in \{A_{i1}, \ldots, A_{iN_i}\}$ and y_{ik} are binary variables such that

$$\sum_{k=1}^{N_i} y_{ik} = x_i. \tag{46}$$

Since x_i is a binary variable for each i, then the last expression is equivalent to

$$\sum_{k=1}^{N_i} y_{ik} \leq 1. \tag{47}$$

So, for each bar i, $A_i x_i$ is zero or assumes the value of only one discrete size in the set S_i. Moreover,

$$A_i v_i^l = \sum_{k=1}^{N_i} A_{ik} q_{ik}^l, \tag{48}$$

where q_{ik}^l is the deformation corresponding to the k-th possible discrete size for the cross-sectional area of the i-th bar under l-th loading condition. The variables q_{ik}^l satisfy the following expressions

$$\sum_{k=1}^{N_i} q_{ik}^l = v_i^l \tag{49}$$

$$d_{min_i} y_{ik} \leq q_{ik}^l \leq d_{max_i} y_{ik}. \tag{50}$$

By using the expressions (45), (46), (48) and (49) and adding the constraints (47) and (50) the following mixed–integer $0-1$ linear program is obtained:

$$\text{(P3)} \qquad \textit{Minimize} \qquad V=\sum_{i=1}^{nb}\left(\sum_{k=1}^{N_i}A_{ik}y_{ik}\right)h_i$$

subject to

$$\sum_{i=1}^{nb}M_{ji}\left(\sum_{k=1}^{N_i}A_{ik}q_{ik}^l\right)-\sum_{m=1}^{na}B_{jm}r_{e_m}^l-f_j^l=0 \tag{51}$$

$$d^l=Cu^l \tag{52}$$

$$u_{min}\le u^l\le u_{max} \tag{53}$$

$$d_{min_i}y_{ik}\le q_{ik}^l\le d_{max_i}y_{ik} \tag{54}$$

$$d_{min_i}\left(1-\sum_{k=1}^{N_i}y_{ik}\right)\le d_i^l-\sum_{k=1}^{N_i}q_{ik}^l\le d_{max_i}\left(1-\sum_{k=1}^{N_i}y_{ik}\right) \tag{55}$$

$$u_{j_m}^l=0 \tag{56}$$

$$-C^Te_p^l+Br_p^l+\lambda f^l=0 \tag{57}$$

$$t_{min_i}\sum_{k=1}^{N_i}A_{ik}y_{ik}\le e_{p_i}^l\le t_{max_i}\sum_{k=1}^{N_i}A_{ik}y_{ik} \tag{58}$$

$$z_n\le\sum_{i\in I(n)}\sum_{k=1}^{N_i}y_{ik}\le|I(n)|\,z_n \tag{59}$$

$$2*\sum_{n=1}^{nn}z_n-\sum_{i=1}^{nb}\sum_{k=1}^{N_i}y_{ik}-\sum_{n=1}^{nn}s_nz_n\le 0 \tag{60}$$

$$-C^Te_a+Br_a+Zf_a=0 \tag{61}$$

$$t_{min_i}\sum_{k=1}^{N_i}A_{ik}y_{ik}\le e_{a_i}\le t_{max_i}\sum_{k=1}^{N_i}A_{ik}y_{ik} \tag{62}$$

$$y_{ik}\in\{0,1\} \tag{63}$$

$$\sum_{k=1}^{N_i}y_{ik}\le 1, \tag{64}$$

where $l=1,\ldots,nc,\ j=1,\ldots,2nn,\ j_m=1,\ldots,na,\ k=1,\ldots,N_i,\ n=1,\ldots,nn$ and $i=1,\ldots,nb$.

This problem is a mixed–integer linear program with some $0-1$ variables with

$$nc\times\left(4nn+5nb+2\sum_{i=1}^{nb}N_i\right)+3nb+4nn+1$$

constraints and

$$nc\times\left(2nb+2nn+2na+\sum_{i=1}^{nb}N_i\right)+\sum_{i=1}^{nb}N_i+nn+nb+na$$

variables. This dimension is considerably larger than the dimension of the problem (P1). However, all the functions involved in this formulation are linear and this enables the use of an integer linear programming code for its solution.

5 Computational Experience and Conclusions

In this section computational experience is reported on the solution of some structural models by using the mixed-integer formulation (P3). These experiences have been performed on a Pentium IV 2.4GHz with 256 MB of RAM. Moreover, the commercial program OSL of the GAMS collection [7] has been used to process the mixed-integer linear programs (P3).

(I) Test Problems

In each test problem the corresponding initial structure consists of nodal points and bars and takes a similar form to the type mesh displayed in Figure 2.

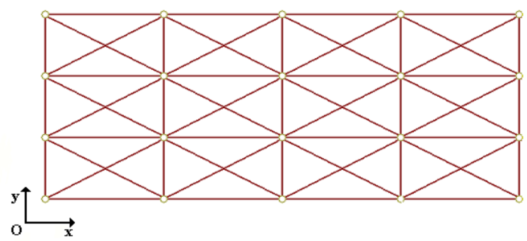

Figure 2: Initial mesh

The main goal of this model is to find the set of included bars in the so–called optimal shape of the structure, which is given by the values of the $0-1$ variables x_i in the optimal solution of the problem.

Different types of sizes of initial meshes, and different applied nodal forces have been taken in consideration in the construction of the test problems. Four sizes of initial meshes, M_i, $i = 0, \ldots, 3$, have been considered, whose topologies are presented in Table 1 leading to five test problems Pt0 to Pt4, according to the following definitions:

- Pt0 - mesh $M0$ and only one nodal load is applied ($f^1_{x_4} = 65$, $f^1_{y_4} = 0$).
- Pt1, Pt2 - mesh $M1$ and two types of applied nodal loads are applied. In Pt1 only one nodal load ($f^1_{x_8} = 0$, $f^1_{y_8} = -65$) is applied, while two nodal loads ($f^1_{x_8} = 0$, $f^1_{y_8} = -65$, $f^2_{x_9} = -40$, $f^2_{y_9} = -40$) are applied in Pt2.
- Pt3 - mesh $M2$ and two nodal loads are simultaneously applied ($f^1_{x_3} = -45.9619$, $f^1_{y_3} = 45.9619$, $f^1_{x_{12}} = -45.9619$, $f^1_{y_{12}} = -45.9619$).
- Pt4 - mesh $M3$ and only one nodal load is applied ($f^1_{x_{23}} = 0$, $f^1_{y_{23}} = -65$).

In these definitions the following parameters are used:

$f^l_{x_n}$ nodal load in (kN) applied in node n in direction Ox for loads combination l;
$f^l_{y_n}$ nodal load in (kN) applied in node n in direction Oy for loads combination l.

	Mesh	h_x	h_y	nal	nb	nn	na	N_i	S_i
Group I	M0	4	3	2×2	6	4	3	1	3
	M1	8	6	3×3	20	9	3	1	3
	M2	6	9	3×4	29	12	8	1	3
	M3	16	12	5×5	72	25	3	1	3
Group II	Sm1	8	6	3×3	20	9	3	2	0.5;3
	Sm2	8	6	3×3	20	9	3	3	0.5;1;2
	Sm3	6	9	3×4	29	12	8	2	0.5;3
	Sm4	6	9	3×4	29	12	8	3	0.5;2;3

Table 1: Test Problems Meshes

In Table 1 the following notations are included.

nal dimension of the mesh in terms of number of nodal in Ox and Oy axes, respectively (in Figure 2, $nal = 5 \times 4$)
h_x total length (in m) to the Ox axis
h_y total length (in m) to the Oy axis
nb number of bars
nn number of nodes
na number of simple supports
S_i set of discrete sizes available for cross-sectional area of bar i (in cm^2)
N_i number of discrete sizes available for cross-sectional area of bar i

In the first group of test problems, structures have been considered for which an unique discrete value is given in each cross-sectional area of each bar. In the second set of problems it is allowed that each bar of the structure assumes one of the values in a finite set of discrete sizes available for its cross-sectional area. This last group leads to four additional test problems, denoted by St1, St2, St3 and St4, and whose associated initial meshes are Sm1, Sm2, Sm3 and Sm4, respectively. The meshes Sm1 and Sm2 have the same dimensions of the M1 mesh, while Sm3 and Sm4 have the same dimensions of the ones in M2. The nodal loads applied in St1 and St2 are the same as in Pt1, while in St3 and St4 are the same as in Pt3. The number of constraints (nr) and the number of variables (nv) of formulations P3 associated to these test problems are presented in the Table 2.

		P3	
	Prob	nr	nv
Group I	Pt0	93	51
	Pt1	273	136
	Pt2	449	220
	Pt3	387	205
	Pt4	921	444
Group II	St1	313	176
	St2	353	216
	St3	445	263
	St4	503	321

Table 2: Dimensions of test problems

In all test problems the displacements and bars stress limits considered are $u_{max} = -u_{min} = 50cm$, $t_{max} = -t_{min} = 355MPa$, respectively and the partial safety factor λ is equal to 1.5.

(II) Numerical Results

Tables 3 and 4 display the numerical results corresponding to the performance of the OSL code for finding an optimal solution of each one of the test problems associated to formulation (P3). This performance is evaluated in terms of number of pivot step iterations (NI), nodes (ND) and CPU time in seconds (T) . In this table, the notation "> 25000000" is used whenever the code has been unable to find an optimal solution for a particular test problem after 25000000 iterations (pivot step iterations). The optimal objective function value (OBJ), in dm^3, found by the solver is also included. In the failure case (NI> 25000000), (OBJ) corresponds to the best upper bound computed by the algorithm. Note that best solution found may be not optimal for the problem (P3) in this last case.

	OSL			
PROB	NI	T	ND	OBJ (dm^3)
PT0	53	0.04	7	3.60
PT1	3033	0.69	311	10.80
PT2	5579	1.77	497	12.90
PT3	891143	325.64	82075	11.92
PT4	>25000000	15018.78	347541	27.30

Table 3: Numerical results with only one available value for the cross-sectional area

	OSL			
PROB	NI	T	ND	OBJ (dm^3)
ST1	64943	22.80	8132	7.05
ST2	57473	30.01	10052	4.90
ST3	4788682	3996.54	411084	6.29
ST4	20606789	61486.08	1496081	5.46

Table 4: Numerical results with more than one available value for the cross-sectional area

The results presented in Tables 3 and 4 show that for meshes of small and average dimension the integer programming code OSL is able to find an optimal solution that leads to structural shapes containing a smaller number of nodes and bars. This is in accordance to the objective of finding an optimal structure with the smallest possible volume. However, for meshes of larger dimensions we may not guarantee that the best solution found is optimal. But, even in this case the corresponding solution leads into a structure of small volume. So the results clearly indicate the validity of the new formulation, as in general the solution of the integer program corresponds to optimal structural shapes that are kinematically stable and involve a small amount of material. This conclusion is well illustrated in Figure 3, which includes the initial mesh M2 and the optimal and deformed forms obtained for the problems PT3, ST3 and ST4 associated to this mesh. The bar forces associated to the bars as well as the elastic and the plastic reactions are indicated under parenthesis. It is not difficult to see that the optimal structure requires a small amount of material and is kinematically stable.

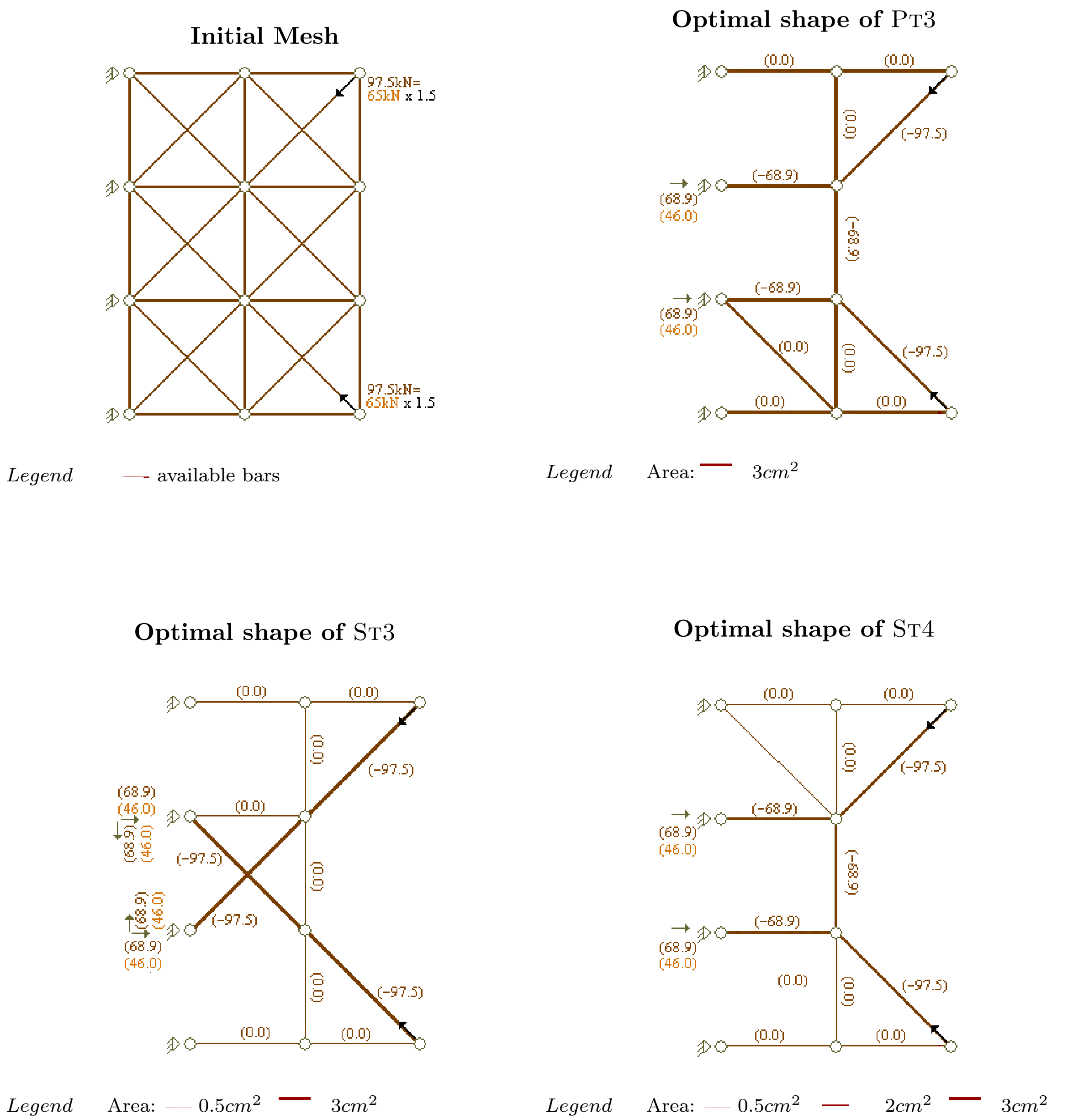

Figure 3: Initial and optimal shape structures of PT3, ST3 and ST4

The numerical results also show that a branch-and-bound algorithm, such as OSL, may face difficulties to find the optimal solution for the integer program corresponding to structures with large number of nodes and bars. An alternative technique for processing the linear integer program in this instance is then required. This algorithm should be designed to find the global minimum for the associated integer program or at least a good feasible solution requiring a small amount of material. The design of such algorithm will certainly be a topic for future research.

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