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On the Solution of NP-hard Linear Complementarity Problems

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Abstract

In this paper two enumerative algorithms for the Linear Complementarity Problems (LCP) are discussed. These procedures exploit the equivalence of the LCP into a nonconvex quadratic and a bilinear programs. It is shown that these algorithms are efficient for processing NP-hard $LCPs$ associated with reformulations of the Knapsack problem and should be recommended to solve difficult $LCPs$.

Key Words: mathematical programming, complementarity, global optimization, enumerative algorithms.

AMS subject classification: 90C33, 65K10.

1 Introduction

The Linear Complementarity Problem (LCP) consists of finding vectors $z \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$ such that

$$\begin{aligned} & w = q + Mz \\ & z \geq 0, \quad w \geq 0 \\ & z^T w = 0 \end{aligned}$$

for a given matrix $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$. This problem has originally appeared in the sixties for the solution of bimatrix games and convex quadratic programs. Since then, it has received an increasing interest,

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mainly due to its many applications in several areas of science, economics and engineering, Cottle et al. (1992) and Murty (1988).

A large number of direct and iterative algorithms has been proposed for the solution of the LCP, Cottle et al. (1992) and Murty (1988). The use of these procedures faces some limitations, as they are only able to process the LCP when the matrix M belongs to some classes of matrices. The so-called Positive Semi-Definite (PSD) and P matrices should be distinguished as the most interesting of these classes. In the first case the LCP has always a solution provided it is feasible, that is, if the linear constraints are consistent, Cottle et al. (1992) . On the other hand the LCP has a unique solution for each vector $q \in \mathbb{R}^n$ when M is a P matrix, Cottle et al. (1992) . In both cases, Lemke's almost complementary, Cottle et al. (1992) and Murty (1988), and interior-point methods, Kojima et al. (1991) and Wright (1997), have proven to be quite successful to process the LCP. In these cases a solution of the LCP can also be found as a stationary point of the following quadratic program, Cottle et al. (1992),

$$\begin{array}{ll} \text{Minimize} & z^T w \\ \text{subject to} & w - Mz = q \\ & z \geq 0, \quad w \geq 0 \end{array} \tag{1.1}$$

As stated in Murty (1988) and Nocedal and Wright (1999), there are a number of efficient algorithms to perform such a task.

In general, a solution of the LCP requires the computation of a global minimum of the quadratic program (1.1). Finding such a point is considered to be a NP-hard problem, Horst et al. (1995). However, the LCP has a solution if and only if there is a global minimum of this program with a value equal to zero. This property is quite important for processing the LCP by a global optimization technique. An enumerative method for the LCP that exploits its quadratic programming formulation (1.1) has been introduced in Al-Khayyal (1987) and subsequently improved, implemented and tested in Júdice and Faustino (1998).

The LCP can also be reformulated as the following bilinear program

$$\begin{array}{ll} \text{Minimize} & q^T x + e^T z + x^T (M - I) z \\ \text{subject to} & Mz \geq -q \\ & z \geq 0 \\ & 0 \leq x \leq e \end{array} \tag{1.2}$$

where I is the identity matrix of order n and $e \in \mathbb{R}^n$ is a vector of ones. There exists a number of techniques for finding a global minimum of such a program, Floudas (2000), Horst et al. (1995) and Sherali and Adams (1999). In particular, a sequential enumerative method has been proposed in Júdice and Faustino (1991) for such a goal. Extensive computational experience reported in Júdice and Faustino (1991) has shown that this algorithm is in general able to find a global minimum in a reasonable amount of time, but faces difficulties to guarantee that such a point has been achieved. As before, the LCP has a solution if and only if a global minimum exists with value equal to zero. This property should be exploited in the sequential enumerative procedure in order to process the LCP.

It is still possible to reduce the LCP into a mixed integer linear program by exploiting the bilinear programming formulation presented before and using the so-called reformulation-linearization technique, Sherali and Adams (1999) and Sherali et al. (1998). Some computational experience reported elsewhere, Sherali et al. (1998), indicates that this formulation may be exploited for processing $LCPs$ with small dimensions. However, the introduction of $O(n^2)$ variables and constraints seems to be a serious drawback for its application to $LCPs$ of larger dimensions.

As the title of this paper indicates, our main motivation is to investigate how NP-hard $LCPs$ should be solved in practice. It is known that Knapsack problems are NP-hard and can be transformed into a $LCPs$ by using simple transformations, Chung (1989), Kojima et al. (1991) and Murty and Júdice (1996). We have tested extensively a number of $LCPs$ associated with Knaspsack problems by using the enumerative methods that are based on the quadratic programming and bilinear programming formulations of the LCP. The results reported in this paper seem to indicate that NP-hard $LCPs$ can be solved in a reasonable amount of time by these techniques. Furthermore traditional direct and iterative methods, such as, Lemke's and interior-point algorithms are unable to process these NP-hard $LCPs$. It should be added that special versions of interior-point algorithms, Conn et al. (2000), Gay et al. (1998) and Vanderbei and Shanno (1999), and a modification of Lemke's method, Murty (1988), have been developed to find stationary points of nonconvex quadratic programs. These procedures are then able to find a stationary point of (1.1). As stated before, such a point does not in general lead to a solution of the LCP, and even these extended algorithms are not appropriate to process the LCP in general.

As expected, the use of a branch-and-bound method to process the integer programming formulation of the LCP is not appropriate, particularly when the dimension of the LCP increases.

The organization of this paper is as follows. In Section 2 the LCP is briefly introduced. The formulations of the LCP mentioned before are discussed in Section 3. The enumerative algorithms are described in Sections 4 and 5. The $LCPs$ associated with the Knapsack problem are presented in Section 6. Computational experience with the enumerative methods is reported in Section 7. Finally some conclusions and hints for future research are presented in the last section of this paper.

2 Formulations of LCP as Global Optimization Problems

As stated in the previous section, given a square matrix M of order n and a vector $q \in \mathbb{R}^n$, the Linear Complementarity Problem, denoted by LCP or $LCP(q, M)$, searches for vectors $z \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$ such that

$$w = q + Mz, \quad z \geq 0, \quad w \geq 0 \tag{2.1}$$

$$z^T w = 0 \tag{2.2}$$

So this problem contains the linear constraints (2.1) that constitute the so-called feasible set K and the nonlinear complementarity condition (2.2). A solution (z, w) is said to be *feasible* if it belongs to the feasible set K. On the other hand it is called *complementary* if it satifies

$$\begin{aligned} & w = q + Mz \\ & z_i w_i = 0 \qquad i = 1, 2, \ldots, n \end{aligned} \tag{2.3}$$

Due to the nonnegative conditions on the variables z_i and w_i, (z, w) is a solution of the LCP if and only if it is feasible and complementary. Furthermore the LCP has no solution if it is infeasible ($K = \emptyset$) or if it is feasible but has no complementary solution in K, that is, $z^T w > 0$ for all $(z, w) \in K$.

Consider now the quadratic program (1.1) introduced in the last section

$$\begin{array}{ll} \text{Minimize} & g_1(z, w) = z^T w \\ \text{subject to} & w - Mz = q \\ & z \geq 0, \quad w \geq 0 \end{array} \tag{QP}$$

As the objective function is bounded from below on its constraint set K, there are three possible cases:

(i) $K = \emptyset$ and the LCP is infeasible.

(ii) $g_1(\bar{z}, \bar{w}) = \min_{(z,w)\in K} g_1(z, w) = 0$ and $(\bar{z}, \bar{w})$ is a solution of the LCP.

(iii) $\min_{(z,w)\in K} g_1(z, w) > 0$ and the LCP is feasible but has no solution.

So the LCP has a solution if and only if it is feasible and there is an optimal solution of the quadratic program with objective function value equal to zero. A global quadratic optimization algorithm, Floudas (2000) and Horst et al. (1995), may become much more efficient for processing $LCPs$, as $g_1(z, w) = 0$ provides a stopping criterium for the procedure. As discussed in the next sections, this feature is essential for the good performance of an enumerative method for solving NP-hard $LCPs$.

Another global optimization formulation for the LCP can be obtained by introducing an additional vector $x \in \mathbb{R}^n$ of $0 - 1$ variables. Since $z_i(q + Mz)_i = 0$ if and only if

$$x_i(q + Mz)_i = 0, \quad (1 - x_i)z_i = 0, \quad x_i \in \{0, 1\} \text{ for all } i = 1, \ldots, n,$$

then the objective function of the QP above can be written as

$$\sum_{i=1}^{n} z_i\, w_i = \sum_{i=1}^{n} z_i(q + Mz)_i = \sum_{i=1}^{n} [(1 - x_i)z_i + x_i(q + Mz)_i]$$

The QP is then equivalent to the following Mixed Integer Bilinear Program

$$\begin{array}{ll} \text{Minimize} & g_2(z, w) = \sum_{i=1}^{n} [(1 - x_i)z_i + x_i(q + Mz)_i] \\ \text{subject to} & Mz \geq -q \\ & z \geq 0 \\ & x_i \in \{0, 1\}, \quad i = 1, 2, \ldots, n \end{array} \qquad \text{(MIBLP)}$$

Since the objective function $g_2(z, w)$ is bilinear and is bounded from below on its constraint set, there exists an optimal solution $(\bar{z}, \bar{x})$ such that $\bar{x}$ is an extreme point of

$$K_x = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1, \quad i = 1, \ldots, n\}$$

$\bar{z}$ is an extreme point of

$$K_z = \{z \in \mathbb{R}^n \; : \; Mz \geq -q, \quad z \geq 0\}$$

provided $K_z \neq \emptyset$, Konno (1976). So the 0-1 constraints presented in $MIBLP$ can be replaced by the continuous constraints that define K_x. The $MIBLP$ is then equivalent to the following Bilinear Program, Mangasarian (1995),

$$\begin{aligned} \text{Minimize} \quad & g_2(z,w) = q^T x + e^T z + x^T (M - I) z \\ \text{subject to} \quad & Mz \geq -q \\ & z \geq 0 \\ & 0 \leq x \leq e \end{aligned} \tag{BLP}$$

where I is the identity matrix of order n and $e \in \mathbb{R}^n$ is a vector of ones. As before, if $K_z \neq \emptyset$, then (z, x) is a global minimum with a zero value if and only if $(z, w = q + Mz)$ is a solution of the LCP.

The LCP can also be reduced into a Mixed Integer Linear Program. In order to get this problem, we first introduce in the constraints of the $MIBLP$ presented before, the complementarity conditions

$$(1 - x_i) z_i = 0, \quad i = 1, 2, \ldots, n$$

to obtain the following equivalent problem

$$\text{Minimize} \quad \sum_{i=1}^{n} q_i x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} x_i z_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} m_{ij} z_j + q_i \geq 0, \qquad i = 1, 2, \ldots, n \tag{2.4}$$

$$z_i \geq 0, \qquad i = 1, 2, \ldots, n \tag{2.5}$$

$$(1 - x_i) z_i = 0, \qquad i = 1, 2, \ldots, n \tag{2.6}$$

$$x_i \in \{0, 1\}, \qquad i = 1, 2, \ldots, n \tag{2.7}$$

The reformulation-linearization technique discussed in Sherali and Adams (1999) and Sherali et al. (1998) is then applied to this problem and consists of the following operations:

(i) Multiply each one of the constraints (2.4) and (2.5) by x_k and $(1 - x_k)$ for all $k = 1, \ldots, n$.

(ii) Replace the products $x_i z_j$ by the variables y_{ij} for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

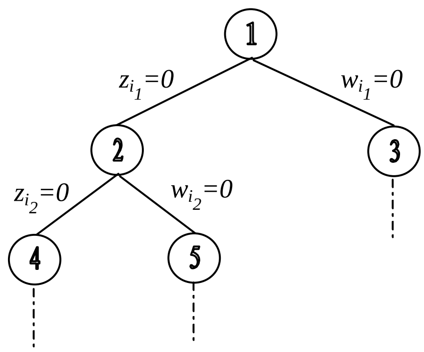
Then the following Mixed Integer Linear Programming Problem is obtained

$$
\begin{array}{lllrcl}
\text{Minimize} & g_3(z,x,y) = \sum_{i=1}^{n} q_i x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij} y_{ij} & & & & \\
\text{subject to} & \sum_{j=1}^{n} m_{ij} y_{kj} + q_i x_k \geq 0, & & k,i & = & 1,2,\ldots,n \\
 & \sum_{j=1}^{n} m_{ij} z_j + q_i \geq \sum_{j=1}^{n} m_{ij} y_{kj} + q_i x_k, & & k,i & = & 1,2,\ldots,n \\
 & 0 \leq y_{ij} \leq z_j, & & i,j & = & 1,2,\ldots,n \\
 & y_{jj} = z_j, & & j & = & 1,2,\ldots,n \\
 & x_i \in \{0,1\}, & & i & = & 1,2,\ldots,n
\end{array} \quad \text{(MILP)}
$$

As is shown in Sherali et al. (1998), $(\bar{x}, \bar{z}, \bar{y})$ is an optimal solution of $MILP$ with $g_3(\bar{x}, \bar{z}, \bar{y}) = 0$ if and only if $(\bar{z}, \bar{w} = q + M\bar{z})$ is a solution of the LCP.

3 An Enumerative Algorithm based on the QP Formulation

This algorithm finds a solution to the LCP by exploring a tree of the form:



In each node of the tree a quadratic program is considered that is obtained from the QP presented in the previous section by adding some constraints $u_i = 0$, where u_i is a complementary z_i or w_i variable. For instance in node

5 of the tree above, the quadratic program takes the form

$$\begin{aligned} \text{minimize } & g_1(z,w) = z^T w \\ \text{subject to } & w - Mz = q \\ & z_{i_1} = 0 \\ & w_{i_2} = 0 \\ & z \geq 0, \quad w \geq 0 \end{aligned}$$

A modified reduced-gradient (MRG) algorithm, Al-Khayyal (1987), is applied in each node in order to find a so-called *local star minimum* of the function g_1 on a set $\bar{K}$ that contains the linear constraints associated to this node. We recall that a local star minimum of g_1 on $\bar{K}$ is an extreme point $(\bar{z},\bar{w})$ of $\bar{K}$ satisfying

$$g_1(\bar{z},\bar{w}) \leq g(z,w)$$

for all adjacent extreme points $(z,w) \in \bar{K}$ of $(\bar{z},\bar{w})$.

In order to describe the MRG algorithm, let $(\bar{z},\bar{w})$ be an extreme point of $\bar{K}$ which corresponds to a basic feasible solution with basis matrix B. If (z^*,w^*) is an adjacent extreme point, then

$$(z^*,w^*) = (\bar{z},\bar{w}) + \mu(d_z,d_w)$$

where μ is the so-called maximum stepsize used in the simplex method and $d = (d_z,d_w)$ is a feasible direction in which d_z and d_w are vectors containing all the components d_i of d associated with the variables z_i and w_i respectively. This feasible direction can be defined in terms of the basis matrix B and of the columns of the matrix M or of the identity matrix I. To show this, let F and T the index sets of the basic and nonbasic variables respectively and let s be the index of the entering nonbasic variable that is increased from zero to generate the adjacent basic feasible solution associated to (z^*,w^*). Then the feasible direction d is given by

$$\begin{aligned} & d_s = 1 \\ & d_j = 0 \text{ for all } j \in T - \{s\} \end{aligned} \tag{3.1}$$

$$d_F = \begin{cases} -B^{-1}M_{.s} & \text{if } s \text{ is a column of a } z_i \text{ variable} \\ B^{-1}e^s & \text{if } s \text{ is a column of } w_i \text{ variable} \end{cases} \tag{3.2}$$

where $M_{.s}$ and e^s are the sth column of the matrices M and I respectively. The value of the quadratic function $g_1(z, w) = z^T w$ at the new extreme point (z^*, w^*) is then

$$g_1(z^*, w^*) = \bar{z}^T \bar{w} + \mu(\bar{z}^T d_w + \bar{w}^T d_z) + \mu^2 d_z^T d_w$$

Therefore there is a decrease on a movement to a new adjacent extreme point ($\mu > 0$) if and only if

$$\bar{z}^T d_w + \bar{w}^T d_z + \mu d_z^T d_w < 0 \tag{3.3}$$

In each iteration of the MRG algorithm, let $(\bar{z}, \bar{w})$ be the current extreme point. The algorithm searches for a feasible descent direction d and a positive stepsize μ satisfying (3.3). If such d and μ exist, the algorithm moves to the new adjacent extreme point with a decrease of the objective function g_1. Otherwise the algorithm terminates with a local star minimum provided all the stepsizes μ are positive. Degenerate cases where $\mu = 0$ can be handled by the so-called Bland's rule, Júdice and Faustino (1988b).

Let

$$\varphi(z, w) = \bar{w}^T \bar{z} + \bar{z}^T w + \bar{w}^T z$$

the linear approximation of $g_1(z, w)$ at the extreme point $(\bar{z}, \bar{w})$. Then it is easy to show. Júdice and Faustino (1988). that $\bar{z}^T d_w + \bar{w}^T d_z$ is the reduced-cost coefficient $\bar{c}_s$ of $\varphi(z, w)$ associated to the nonbasic entering variable z_s or w_s that is increased to generate the new adjacent extreme point. So, if $d_z^T d_w \leq 0$ then $\bar{c}_s < 0$ implies that

$$g_1(z^*, w^*) \leq g(\bar{z}, \bar{w}) + \bar{c}_s \mu < g(\bar{z}, \bar{w})$$

provided $\mu > 0$. So, as in the simplex method for linear programming a negative reduced-cost coefficient means a descent feasible direction. Therefore if there is a guarantee that $d_z^T d_w \leq 0$ always holds, then the MRG reduces to the usual simplex method.

The steps of the enumerative method are presented below.

Enumerative QP Algorithm

Step 0 Let $L = \{1\}$ be the initial list of open nodes and $QP(1)$ be the quadratic program QP introduced in the previous section.

Step 1 If $L = \emptyset$, stop: LCP has no solution. Otherwise choose a node $t \in L$.

Step 2 Remove t from L. Apply the MRG algorithm to the quadratic program $QP(t)$ associated to the node t. If $QP(t)$ is infeasible, go to Step 1. Otherwise compute a local star minimum $(\bar{z}, \bar{w})$ for $QP(t)$.

Step 3 If $g_1(\bar{z}, \bar{w}) = \bar{z}^T \bar{w} = 0$, stop: $(\bar{z}, \bar{w})$ is a solution of the LCP. Otherwise let (z_r, w_r) be a pair of positive basic variables in the local star minimum $(\bar{z}, \bar{w})$.

Step 4 Add two new nodes k and $(k+1)$ to the list L, with quadratic programs $QP(k)$ and $QP(k+1)$ defined by:

$$\begin{aligned} QP(k) &: QP(t) \text{ and constraint } z_r = 0 \\ QP(k+1) &: QP(t) \text{ and constraint } w_r = 0 \end{aligned}$$

Go to Step 1.

It follows from the description of the algorithm that only basic solutions of the LCP are used throughout the algorithm. Hence the enumerative method can be implemented for large-scale $LCPs$ by exploiting reinversion and updating schemes for sparse LU factorization of the basis matrices, Murty (1983). Such an implementation should also contain some heuristics techniques for choosing the pair of variables for searching in Step 3 and the node of the list L in Step 1. We suggest Júdice and Faustino (1988a) for a detailed description of this implementation.

4 An Enumerative Sequential Algorithm based on the BLP Formulation

Consider again the BLP formulation of the LCP introduced in Section 3. We can write this problem in the form

$$\begin{aligned} \text{Minimize } & q^T x + min\left\{(e + (M^T - I)x)^T z : \ Mz \geq -q, \ z \geq 0\right\} \\ & 0 \leq x \leq e \end{aligned}$$

The dual of the inner linear program takes the form:

$$\begin{aligned} \text{Minimize} \quad & -q^T u \\ \text{subject to} \quad & M^T u \leq e + (M^T - I)x \\ & u \geq 0 \end{aligned}$$

By using the complementarity slackness property, Murty (1983), it is then possible to reduce the BLP into the following Mathematical Programming with Equilibrium (or Complementarity) Constraints

$$\begin{array}{ll} \text{Minimize} & q^T x - q^T u \\ \text{subject to} & \begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} e \\ q \end{bmatrix} + \begin{bmatrix} 0 & -M^T \\ M & 0 \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix} + \begin{bmatrix} M^T - I \\ 0 \end{bmatrix} x \\ & y, v, z, u \geq 0 \\ & 0 \leq x \leq e \\ & \begin{bmatrix} z \\ u \end{bmatrix}^T \begin{bmatrix} y \\ v \end{bmatrix} = 0 \end{array} \tag{MPEC}$$

The constraints of this $MPEC$ constitute a General Linear Complementarity Problem ($GLCP$) in which the matrix associated with the complementary variables (z, u) is PSD. As stated in Fernandes et al. (2001) and Júdice and Vicente (1994), this $GLCP$ can be processed by a number of techniques. Among them, the MRG algorithm described in the previous section can be easily modified to process this problem. Furthermore it is possible to show that each feasible direction

$$d = (d_z, d_u, d_x, d_y, d_v)$$

satisfies

$$d_z^T d_y + d_u^T d_v = 0$$

So the MRG reduces into a simplex type method and terminates with a stationary point of the complementarity function

$$h(z, u, y, v) = z^T y + u^T v$$

on the set of the linear constraints of the $GLCP$. As is shown in Fernandes et al. (2001), this stationary point is a solution of the $GLCP$. We have then shown that the MRG algorithm is an efficient procedure for finding a solution of this $GLCP$.

Suppose that a solution $(\bar{z}, \bar{u}, \bar{y}, \bar{v}, \bar{x})$ of the $GLCP$ has been found. Then it may be possible to reduce further the value of the objective function by the Basis Restricted Simplex (BRS) method described in Bialas and Karwan (1984) and Júdice and Faustino (1991). This algorithm terminates in a new solution $(\widetilde{z}, \widetilde{u}, \widetilde{y}, \widetilde{v}, \widetilde{x})$ of the $GLCP$ in a finite number of pivot steps. Now two cases may occur and are stated below.

(i) If $q^T\widetilde{x} - q^T\widetilde{u} = 0$, then $(\widetilde{z}, \widetilde{w} = q + M\widetilde{z})$ is a solution of the LCP.

(ii) If $q^T\widetilde{x} - q^T\widetilde{u} > 0$, then set

$$\lambda = q^T\widetilde{x} - q^T\widetilde{u} - \epsilon(q^T\widetilde{x} - q^T\widetilde{u}) \tag{4.1}$$

where ϵ is a small positive number. Consider the $GLCP(\lambda)$ obtained by the previous $GLCP$ by adding the constraint

$$q^T x - q^T u \leq \lambda$$

Then $(\widetilde{z}, \widetilde{u}, \widetilde{y}, \widetilde{v}, \widetilde{x})$ is infeasible for this $GLCP(\lambda)$. The enumerative method described in the previous section can easily be adapted to solve this new $GLCP(\lambda)$ and one of the three following cases should occur:

(i) $GLCP(\lambda)$ has no solution and the same happens to the LCP.

(ii) $GLCP(\lambda)$ has a solution $(\bar{z}, \bar{u}, \bar{y}, \bar{v}, \bar{x})$ with $q^T\bar{x} - q^T\bar{u} = 0$ and $(\bar{z}, \bar{w} = q + M\bar{z})$ is a solution of the LCP.

(iii) $GLCP(\lambda)$ has a solution $(\bar{z}, \bar{u}, \bar{y}, \bar{v}, \bar{x})$ with $q^T\bar{x} - q^T\bar{u} > 0$. Then the BRS algorithm is applied starting from this basic feasible solution and a new solution $(\widetilde{z}, \widetilde{u}, \widetilde{y}, \widetilde{v}, \widetilde{x})$ of the $GLCP(\lambda)$ is obtained. Now, either $q^T\widetilde{x} - q^T\widetilde{u} = 0$ and $(\widetilde{z}, \widetilde{w} = q + M\widetilde{z})$ is a solution of the LCP or a new $GLCP(\lambda)$ has to be solved, where λ is previously updated by (4.1).

It follows from the description of this algorithm, that a finite sequence of $GLCPs$ has to be solved in order to find a solution of the LCP or to show that none exists. The enumerative method described in the previous section is easily generalized to process all these $GLCPs$ and can be implemented along the same lines for dealing with large-scale $LCPs$. A drawback of this approach lies on the fact that the dimension of the $GLCPs$ is twice that of the LCP. However, the $GLCPs$ possess a quite nice structure, the matrix of the (z, u) variables is PSD and the MRG algorithm can in many cases reduce to simplex type method. As is reported in the last section of this paper, the enumerative BLP algorithm requires in some cases a small number of pivot steps to process the LCP than the enumerative QP method.

5 Reduction of the Knapsack Problem into a LCP

Given a positive real number b and a positive vector $a \in \mathbb{R}^n$, The Knapsack Problem consists of finding a vector $x \in \mathbb{R}^n$ such that

$$\begin{aligned} & a^T x = b \\ & x_i \in \{0,1\}, \quad i = 1,2,\ldots,n \end{aligned} \tag{5.1}$$

In this section three different formulations of the Knapsack problem as an LCP are introduced. These formulations rely on the fact that for each $i = 1,2,\ldots,n$, $x_i \in \{0,1\}$ is equivalent to

$$w_i = 1 - x_i, \quad x_i \geq 0, \quad w_i \geq 0, \quad x_i w_i = 0$$

Based on this reduction, it is easy to show Chung (1989) that the Knapsack problem is equivalent to the following LCP of dimension $(n+2)$:

$$\begin{array}{llll} w & = & e - x & \\ w_{n+1} & = & -b + a^T x - \alpha x_{n+1} & \\ w_{n+2} & = & b - a^T x & -\beta x_{n+2} \\ x_i \geq 0, & w_i \geq 0, & x_i w_i = 0, & i = 1,2,\ldots,n+2 \end{array}$$

where $e \in \mathbb{R}^n$ is a vector of ones and α, β are two positive real numbers. The matrix M of this LCP takes then the form:

$$M = \begin{bmatrix} -I & 0 & 0 \\ a^T & -\alpha & 0 \\ -a^T & 0 & -\beta \end{bmatrix} \in \mathbb{R}^{(n+2)\times(n+2)}$$

where I is the identity matrix of order n. Hence this matrix is negative semi-definite(NSD), that is, $-M \in PSD$, if α and β are chosen to satisfy the following inequalities

$$\alpha > \theta \, \frac{a^T a}{4}, \qquad \beta > \theta\alpha \, \frac{a^T a}{4\alpha - a^T a}$$

for $\theta > 1$ a fixed number. On the other hand M is an indefinite (IND) matrix, that is, $M \notin PSD$ and $M \notin NSD$ provided

$$\alpha > \theta \, \frac{a^T a}{4}, \qquad \beta < \frac{\alpha}{\theta} \, \frac{a^T a}{4\alpha - a^T a}$$

where $\theta > 1$ as before.

As explained in Murty and Júdice (1996), the Knapsack problem can also be reduced into the $LCP(q, M)$, where $q = [a, -b, b]^T \in \mathbb{R}^{n+2}$ and

$$M = \begin{bmatrix} I & e & -e \\ e^T & -2n & 0 \\ -e^T & 0 & -2n \end{bmatrix} \in \mathbb{R}^{(n+2)\times(n+2)}$$

where I is the identity matrix of order n and $e \in \mathbb{R}^n$ is a vector of ones. The matrix M is in this case symmetric NSD.

Finally in Kojima et al. (1991), the Knapsack problem has shown to be equivalent to a $LCP(q, M)$, where

(i) the vector q takes the form

$$q = [p \;\; p \;\; \ldots \;\; p \;\; -b \;\; b]^T \in \mathbb{R}^{4n+2}$$

with $p = [0 \;\; 0 \;\; -1 \;\; 1]^T \in \mathbb{R}^4$.

(ii) the matrix M is given by

$$M = \begin{bmatrix} B & 0 & \cdots & 0 & 0 & 0 \\ 0 & B & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & B & 0 & 0 \\ & & \bar{a}^T & & & \\ & & -\bar{a}^T & & & \end{bmatrix} \in \mathbb{R}^{(4n+2)\times(4n+2)}$$

where

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

and the components $\bar{a}_i$ of $\bar{a} \in \mathbb{R}^{4n+2}$ satisfy

$$\bar{a}_i = \begin{cases} a_j, & \text{if} \quad i = 4j-3, \qquad j = 1, 2, \ldots, n \\ 0, & \text{otherwise} \end{cases}$$

As discussed in Kojima et al. (1991), the matrix M of the LCP belongs to the class P_0 of matrices with nonnegative principal minors.

We denote by $FORM2$ and $FORM3$ those LCP formulations of the Knapsack problem in which the associated matrix is symmetric NSD and P_0 respectively. Furthermore the first formulation of the Knapsack problem is denoted by $FORM1\,NSD$ or $FORM1\,IND$, depending on the associated matrix M to be NSD or IND respectively.

It is important to add that all these formulations of the Knapsack problem as $LCPs$ have been established in order to study the computational complexity of the LCP. Since the Knapsack problem is NP-hard, then the same happens to the LCP when its matrix M is symmetric or unsymmetric NSD, IND or P_0. Furthermore it is known that a LCP with a PSD matrix can be solved in polynomial time, Kojima et al. (1991) and Murty (1988), and the complexity of the solution of a LCP with a P matrix is still an open question.

6 Computational Experience

In this section we report some computational experiences with the enumerative methods discussed in Sections 4 and 5 on the solution of NP-hard $LCPs$ associated with Knapsack problems. These experiences have been performed on a Pentium II 350 MHz with 256 MB of RAM. The test problems have been generated from a Knapsack problems in which all the components a_i of the vector $a \in \mathbb{R}^n$ have been randomly generated in the interval $[1, 50]$. The real number b has been set equal to

$$b = \sum_{i \in I} a_i$$

where I is a subset of $\{1, \ldots, n\}$ corresponding to the variables x_i that are equal to one in a solution of the Knapsack problem. Three different sets I have been considered in our experiences that differ on its cardinal to be $\frac{n}{4}$, $\frac{n}{2}$ and $\frac{3n}{4}$, that is, on the percentage of variables equal to one to be 25%, 50% and 75% respectively.

Table 1 displays the results of the performances of the two enumerative methods that are based on the QP ($ENQP$) and BLP ($ENBLP$) formulations of the LCP. In this table $FORM1\,NSD$, $FORM1\,IND$, $FORM2$

and $FORM3$ represent the LCP formulations of the Knapsack problem discussed in the previous section. Furthermore N denotes the dimension of the LCP, NI, ND are the total number of pivot steps, iterations and nodes required by the enumerative algorithms and T is the total CPU time in seconds for solving the LCP. Finally $NGLCP$ corresponds to the number of $GLCPs$ (including the first that does not contain the constraint in λ) that have been processed by the algorithm $ENBLP$.

The results displayed in Table 1 show that the enumerative algorithm $ENQP$ based on the QP formulation has been able to process all the NP-hard LCPs in a reasonable amount of time. Furthermore this algorithm performs in general better than the alternative enumerative algorithm $ENBLP$ that is based on the BLP formulation. However, this latter procedure also performs well in general. A nice feature of the performance of this latter technique $ENBLP$ is the consistently small number of $GLCPs$ that have to be processed in order to solve the LCP.

Table 1 also includes the results of the performance of the branch-and-bound method $ENMILP$ (OSL code, IBM Corporation (1992)) for processing the $MILP$ formulation of the LCP discussed in Section 3. In this experience we have set a limit of 50000 pivot steps and 14400 CPU seconds for the execution of this program. We have written > 50000 and > 14400 in the columns NI and T whenever the corresponding limit has been achieved without terminating the execution of the program. The numerical results clearly indicate that a branch-and-bound methodology is not appropriate for processing the $MILP$ associated with the LCP, particularly when the dimension of the LCP increases. This fact seems to have been noticed by the authors of the $MILP$ formulation of the LCP, as they recommend a subgradient optimization algorithm for processing this $MILP$, Sherali et al. (1998).

It is also important to add that we have been trying to process all the $LCPs$ by two well-known direct and iterative methods, namely Lemke's method and an interior-point algorithm using the codes described in Júdice and Faustino (1988a) and Júdice et al. (1996) respectively. These two algorithms have never been able to solve these $LCPs$. The same conclusions should be achieved for the extensions of these algorithms to find stationary points of noncovex programs discussed in Conn et al. (2000), Gay et al. (1998), Murty (1988) and Vanderbei and Shanno (1999).

This interesting conclusion indicates that there exists a clear gap be-

PROBLEMS	N	ENQP			ENBLP				ENMILP	
		NI	T	ND	$NGLCP$	NI	T	ND	NI	T
FORM1-NSD 25%	22	17	0,06	5	4	338	0,56	99	3430	9,61
	52	237	0,50	55	7	848	1,44	370	4391	71,90
	102	38	0,16	4	7	2481	3,73	554	13971	1028,87
	152	113	0,45	23	4	1231	2,91	317	22043	3372,81
FORM1-NSD 50%	22	21	0,05	5	5	60	0,05	12	626	3,52
	52	150	0,43	35	5	659	0,94	135	7583	156,43
	102	104	0,38	16	4	133	0,50	14	29056	3607,67
	152	174	0,61	32	5	3221	6,88	764	43960	7164,21
FORM1-NSD 75%	22	140	0,22	31	4	343	0,6	89	631	3,84
	52	98	0,16	14	6	3765	4,09	1117	3236	70,30
	102	272	0,72	66	5	1058	1,92	275	20209	3036,23
	152	570	1,49	156	5	1711	3,88	455	>50000	10965,49
FORM1-IND 25%	22	17	0,00	5	4	336	0,55	99	1363	4,06
	52	237	0,50	55	7	848	1,31	370	2065	31,26
	102	38	0,11	4	7	2479	3,90	554	26546	2080,25
	152	113	0,50	23	4	1231	2,68	317	21519	4328,45
FORM1-IND 50%	22	21	0,00	5	5	59	0,11	12	984	2,47
	52	150	0,44	35	5	659	0,89	135	2841	33,45
	102	104	0,33	16	4	133	0,38	14	8870	745,40
	152	174	0,67	32	5	3119	6,86	764	24338	13896,48
FORM1-IND 75%	22	140	0,16	31	4	343	0,6	89	1918	5,27
	52	98	0,17	14	6	3765	4,22	1117	4702	87,61
	102	272	0,54	66	5	1058	1,57	275	9741	1029,75
	152	570	1,37	156	5	1711	4,00	455	>50000	13330,47
FORM2 25%	22	35	0,06	5	5	1230	0,99	369	780	4,78
	52	39	0,00	5	4	1236	1,26	155	9916	462,69
	102	37	0,17	2	3	773	1,32	102	37648	9127,08
	152	56	0,22	2	3	1898	4,57	420	>50000	>14400,00
FORM2 50%	22	343	0,39	46	6	634	0,71	125	506	3,19
	52	78	0,16	8	3	431	0,71	74	12366	455,72
	102	51	0,00	1	4	178	0,66	11	>50000	11811,39
	152	83	0,33	1	7	1347	3,02	203	>50000	>14400,00
FORM2 75%	22	23	0,00	2	2	106	0,11	20	1249	9,34
	52	58	0,05	3	2	232	0,27	46	25875	885,62
	102	81	0,50	1	3	861	1,70	184	>50000	7571,64
	152	117	0,39	2	6	1530	5,34	551	11000	>14400,00
FORM3 25%	22	11	0,00	1	1	35	0,00	1	12	0,17
	50	20	0,00	1	1	73	0,00	1	2688	24,33
	102	153	0,22	7	2	177	0,61	8	22964	1082,20
	150	666	0,82	24	3	11614	16,30	1494	>50000	6570,74
FORM3 50%	22	25	0,00	4	3	155	0,17	34	404	0,66
	50	122	0,00	8	3	4678	3,35	1092	3506	33,17
	102	897	0,70	89	3	1762	1,82	143	5935	219,15
	150	763	0,72	37	4	876	1,42	48	>50000	9788,54
FORM3 75%	22	40	0,00	5	2	277	0,49	57	615	1,15
	50	212	0,28	17	7	1138	1,04	170	4356	42,13
	102	687	0,71	19	3	428	0,82	28	27381	1444,32
	150	1996	1,98	33	4	1824	2,25	90	>50000	6875,35

Table 1: *Performance*

tween the complexity of these $LCPs$ and of $LCPs$ with P and PSD matrices.

7 Conclusions

In this paper we have investigated the solution of NP-hard $LCPs$ associated with Knapsack problems. This study has indicated that direct and iterative methods such as interior-point or Lemke's methods are not appropriate to process these $LCPs$. Special versions of these algorithms, Conn et al. (2000), Gay et al. (1998), Murty (1988) and Vanderbei and Shanno (1999), can successfully be used for computing stationary points of the quadratic program (1.1) that has been associated to the LCP. However such points are not in general solutions of the LCP. On the other hand, two enumerative methods based on quadratic and bilinear programming formulations of the LCP have shown to perform well and should be recommended to process difficult linear complementarity problems taken from Knapsack problems or any other source.

We believe that both the enumerative techniques discussed in this paper can still be improved in order to process more efficiently NP-hard $LCPs$. An implementation of the enumerative QP algorithm based on an active-set methodology may become more efficient than the one discussed in this paper that relies on simplex pivot steps. On the other hand active-set and augmented lagrangian techniques may be appropriate to process the $GLCPs(\lambda)$ that are required by the enumerative BLP algorithm. These two topics will certainly deserve our attention in the next future.

Appendix - List of Abbreviations

Classes of Matrices:

PSD (Positive Semi–Definite): $A \in \text{PSD} \Leftrightarrow x^T Ax \geq 0$ for all x.

NSD (Negative Semi–Definite): $A \in \text{NSD} \Leftrightarrow -A \in \text{PSD}$.

IND (Indefinite): $A \in \text{IND} \Leftrightarrow A \notin \text{PSD}$ and $A \notin \text{NSD}$.

P($\mathbf{P_0}$) $A \in \text{P}(\text{P}_0) \Leftrightarrow$ all principal minors of A are positive (nonnegative).

Optimization Problems:

LCP: Linear Complementarity Problem.

GLCP: General Linear Complementarity Problem.

QP: Quadratic Programming.

BLP: Bilinear Programming.

MIBLP: Mixed Integer Bilinear Programming.

MILP: Mixed Integer Linear Programming.

MPEC: Mathematical Programming with Equilibrium Constraints.

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