

On the Asymmetric Eigenvalue Complementarity Problem

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Abstract

In this paper, we discuss the Eigenvalue Complementarity Problem (EiCP) where at least one of its defining matrices is asymmetric. A sufficient condition for the existence of a solution to the EiCP is established. The EiCP is shown to be equivalent to finding a global minimum of an appropriate merit function on a convex set Ω defined by linear constraints. A sufficient condition for a stationary point of this function on Ω to be a solution of the EiCP is presented. A branch-and-bound procedure is developed for finding a global minimum of this merit function on Ω . In addition, a sequential enumerative algorithm for the computation of the minimum and the maximum eigenvalues is also discussed. Computational experience is included to highlight the efficiency and efficacy of the proposed methodologies to solve the asymmetric EiCP.

Key Words: eigenvalue complementarity, extreme eigenvalues, branch-and-bound, global optimization, nonlinear programming.

1 Introduction

Given a matrix $A \in \mathbb{R}^{n \times n}$ and a positive definite (PD) matrix $B \in \mathbb{R}^{n \times n}$, the Eigenvalue Complementarity Problem (EiCP) consists of finding a real number $\lambda > 0$ and a vector $x \in \mathbb{R}^n \setminus \{0\}$

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such that

$$w = (\lambda B - A)x \quad (1.1)$$

$$w \geq 0, x \geq 0 \quad (1.2)$$

$$x^T w = 0, \quad (1.3)$$

where $w \in \mathbb{R}^n$. The EiCP is a particular case of the following Mixed Eigenvalue Complementarity Problem (MEiCP) introduced in [8]:

$$w = (\lambda B - A)x \quad (1.4)$$

$$w_J \geq 0, x_J \geq 0 \quad (1.5)$$

$$w_J^T x_J = 0 \quad (1.6)$$

$$w_{\bar{J}} = 0, \quad (1.7)$$

where $x \in \mathbb{R}^n \setminus \{0\}$, $w \in \mathbb{R}^n$, $J \subseteq \{1, \dots, n\}$, and $\bar{J} = \{1, \dots, n\} \setminus J$. This MEiCP first appeared in the study of static equilibrium states of finite dimensional mechanical systems with unilateral frictional contact [7, 8]. If $J = \emptyset$, then the MEiCP reduces to the well-known Generalized Eigenvalue Problem, which lends the name to the problem. Any vector solution $x \neq 0$ of the MEiCP (or EiCP) is called a *Complementary Eigenvector* and λ is a *Complementary Eigenvalue*. Some other applications of this type of problem and extensions can be found in [32, 33, 34, 36]. In this paper, we only address the simple EiCP, where the complementary eigenvectors are nonnegative. Most of the main results to be presented in this paper can be easily extended to the MEiCP. As discussed in [30, 21], if A and B are both symmetric matrices, then the EiCP reduces to finding a stationary point of an appropriate merit function on a specially structured convex set. However, in general, at least one of these matrices is asymmetric and different approaches need to be developed for processing the EiCP.

A first approach to solve the asymmetric EiCP is to transform it into a Nonlinear Complementarity Problem (NCP) with $n + 1$ complementary variables [8]. Despite the theoretical and even practical interest of this reduction, the mapping to the NCP is nonmonotone and this has prevented the use of the most efficient complementary solvers [13] to solve the general EiCP. Among these techniques, PATH [12] is capable of processing the largest class of NCPs. Computational experience presented in [30] and in this paper shows that even PATH is not, in general, able to solve the EiCP. Another interesting formulation of the EiCP is to reduce it to a fixed point problem [9]. A gradient algorithm to find such a fixed-point has been introduced in [9], but, as before, this technique is not able to always compute a solution to the EiCP.

In [22], the EiCP has been formulated as a Mathematical Programming Problem with Equilibrium Constraints (MPEC). The nonconvexity of the objective function of the MPEC has prevented the use of efficient methods [5, 17, 18, 20, 23], for finding a global minimum to this problem. There are a number of efficient algorithms for finding stationary points of MPEC [2, 11, 14, 15, 24, 26]. However, there is no guarantee that such stationary points are solutions of the EiCP. For instance, computational experience included in [22] has shown that a stationary point for this MPEC can be found quite efficiently by a Complementarity Active-Set algorithm [24]. As before, the algorithm cannot always guarantee a solution to the EiCP.

Based on these efforts, a branch-and-bound algorithm for the EiCP was introduced in [22]. The algorithm is able, at least in theory, to generally solve the EiCP, but it is not too efficient

in practice. It exploits a nonlinear programming (NLP) formulation of the EiCP and seeks a global minimum for this optimization problem. An interesting property of this NLP is that its global optimal value is zero provided that the EiCP has a solution. This property is exploited in the design of a new enumerative algorithm to be introduced in this paper. This algorithm can also be seen as an extension to the EiCP of an enumerative method that has proved efficient to process solvable Linear Complementarity Problems [1, 19]. A binary tree is explored based on the dichotomy for the complementary variables $x_i = 0$ or $w_i = 0$, $\forall i \in \{1, \dots, n\}$. At each node of the tree, a stationary point of the so-called *complementarity function* $x^T w$ is found on a set defined by the original linear constraints along with some equalities related to the complementary variables that are fixed to zero on the branches connecting the given node to the root node of the tree. For the EiCP, the enumerative method also considers branchings associated with bisections of intervals containing the variable $x_{n+1} = \lambda^{-1}$, where λ is the complementary eigenvalue to be computed by the algorithm. Computational experience with a number of EiCPs has shown that the algorithm is quite efficient in finding a solution to solvable EiCPs and compares favorably with the branch-and-bound method introduced in [22] as well as with the popular commercial software BARON [31] for global optimization.

The design of such an enumerative algorithm has raised the question whether a stationary point of the merit function of the NLP formulation of the EiCP provides a solution to the EiCP. A necessary and sufficient condition is established in the present paper, which shows that this property is related to the values of the Lagrange multipliers associated with two of the linear constraints of the NLP. Another sufficient condition for the existence of a solution to the EiCP is also derived when the matrix A is copositive and satisfies a certain R_0 -property that is defined in [10]. In particular, an EiCP has a solution provided that the matrix A is strictly copositive, a class of matrices that has been quite useful in complementarity problems [10, 13] and global optimization [3].

We also investigate a new sequential algorithm to find the minimum and maximum complementary eigenvalues in a given interval $[l, u]$. The algorithm employs a sequential enumerative method for a finite number of nested intervals $[l_k, u_k]$, where $[l_{k+1}, u_{k+1}] \subset [l_k, u_k] \subset [l, u]$ for each k , until it discovers an unsolvable EiCP. A certificate for the nonexistence of a solution to the last EiCP is provided by the algorithm discussed in [22] and leads to the conclusion that the last computed λ is the minimum or maximum complementary eigenvalue for the EiCP in the interval $[l, u]$. Computational experience reported in this paper shows that this sequential enumerative algorithm is able to efficiently perform this task.

The remainder of this paper is organized as follows. In Section 2, the existence of a solution to the EiCP is established. The NLP formulation of the EiCP and the study of stationary points of its objective function are discussed in Section 3. The enumerative algorithm is introduced in Section 4. The use of this algorithm to find the minimum and maximum complementary eigenvalues is addressed in Section 5. Computational experience with this algorithm for solving EiCPs and computing extremal complementary eigenvalues are reported in Section 6 along with some concluding remarks.

2 Existence of a Solution to the Asymmetric EiCP

In this section, we provide a sufficient condition for the existence of a solution to the EiCP. To do this, we first show that the EiCP is equivalent to a finite dimensional variational inequality [13] on the simplex

$$\Omega = \{x \in \mathbb{R}^n : e^T x = 1, x \geq 0\}, \quad (2.1)$$

where $e \in \mathbb{R}^n$ is a vector of ones. Let $F : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n$ be the mapping defined by

$$F(x) = \left(\frac{x^T A x}{x^T B x} B - A \right) x, \quad (2.2)$$

where we assume that B is a strictly copositive (SC) matrix, that is, it satisfies

$$x^T B x > 0, \forall 0 \neq x \geq 0. \quad (2.3)$$

Note that we have relaxed the PD assumption on the matrix B , as any PD matrix is also SC. As discussed in [13], the variational inequality $VI(F, \Omega)$ consists of finding an $\bar{x} \in \Omega$ such that

$$F(\bar{x})^T (x - \bar{x}) \geq 0, \forall x \in \Omega. \quad (2.4)$$

Next, we establish that the EiCP and the $VI(F, \Omega)$ are equivalent problems.

Theorem 2.1 *If Ω and F are given by (2.1) and (2.2), respectively, then \bar{x} is a solution of $VI(F, \Omega)$ if and only if \bar{x} satisfies the complementarity problem*

$$w = (\lambda B - A)x \quad (2.5)$$

$$w \geq 0, x \geq 0 \quad (2.6)$$

$$e^T x = 1 \quad (2.7)$$

$$x^T w = 0 \quad (2.8)$$

$$\text{with } \bar{\lambda} = \frac{\bar{x}^T A \bar{x}}{\bar{x}^T B \bar{x}}.$$

Proof: \bar{x} is a solution of $VI(F, \Omega)$ if and only if \bar{x} is global minimum of the linear program (LP):

$$\begin{aligned} & \text{Minimize} && F(\bar{x})^T (x - \bar{x}) \\ & \text{subject to} && e^T x = 1 \\ & && x \geq 0. \end{aligned}$$

Consider the dual to this LP:

$$\begin{aligned} & \text{Maximize} && u_0 \\ & \text{subject to} && F(\bar{x}) = u_0 e + w \\ & && w \geq 0. \end{aligned}$$

By the duality theory of linear programming, \bar{x} is a solution of $VI(F, \Omega)$ if and only if it satisfies the conditions

$$F(\bar{x}) = u_0 e + w \quad (2.9)$$

$$\bar{x} \geq 0, w \geq 0$$

$$\bar{x}^T w = 0$$

$$e^T \bar{x} = 1.$$

In order to prove the theorem, we first show that $u_0 = 0$ in (2.9). Note that

$$\bar{x}^T F(\bar{x}) = u_0 e^T \bar{x} + \bar{x}^T w,$$

which implies that

$$\bar{x}^T F(\bar{x}) = u_0.$$

But

$$\bar{x}^T F(\bar{x}) = \bar{x}^T \left(\frac{\bar{x}^T A \bar{x}}{\bar{x}^T B \bar{x}} B - A \right) \bar{x} = 0,$$

and $u_0 = 0$. Thus, with $\lambda = \bar{x}^T A \bar{x} / \bar{x}^T B \bar{x}$, \bar{x} is a solution to VI(F, Ω) if and only if \bar{x} is a solution to (2.5)-(2.8). \square

Since Ω is a convex and compact set, then VI(F, Ω) has a solution [13], a result first established in [25]. So conditions (2.5)-(2.8) are always satisfied whenever B is an SC matrix. In order to guarantee a solution to the EiCP satisfying $\lambda > 0$, we need a further condition on the matrix A . To accomplish this, we recall the following definitions (see [10] for instance):

- (i) A is a copositive matrix ($A \in C$) if and only if $x^T A x \geq 0$ for all $x \geq 0$.
- (ii) A is an R_0 matrix ($A \in R_0$) if and only if $(x \geq 0, Ax \geq 0, x^T A x = 0) \Rightarrow x = 0$.

Note also that any strictly copositive matrix is also copositive. We now establish our existence result for the EiCP.

Theorem 2.2 *If $B \in SC$, $A \in C$, and $-A \in R_0$, then the EiCP has a solution \bar{x} with $\lambda = (\bar{x}^T A \bar{x}) / (\bar{x}^T B \bar{x}) > 0$.*

Proof: Because $A \in C$ and $B \in SC$, there exists an $\bar{x} \in K$ that satisfies (2.5)-(2.8) by the previous theorem, and such that

$$\lambda = \frac{\bar{x}^T A \bar{x}}{\bar{x}^T B \bar{x}} \geq 0.$$

Now, if $\lambda = 0$ then $\bar{x}^T A \bar{x} = 0$ and the system (2.5)-(2.8) reduces to

$$\begin{aligned} w &= -A\bar{x} \\ w &\geq 0, \bar{x} \geq 0 \\ \bar{x}^T w &= 0, \end{aligned}$$

which contradicts that $-A \in R_0$. \square

Note that the theorem is also true if B is a PD matrix, that is, it satisfies

$$x^T B x > 0, \forall x \neq 0,$$

because any PD matrix is also SC. Furthermore, any SC matrix A satisfies $A \in C$ and $-A \in R_0$. Therefore the EiCP has a solution with $\lambda > 0$ if A is a strictly copositive matrix.

In [30], a necessary and sufficient condition for the existence of a solution to the symmetric EiCP is derived. This result states that if A and B are both symmetric matrices, then the EiCP

has a solution with $\lambda > 0$ if and only if there exists $0 \neq \bar{x} \geq 0$ such that $\bar{x}^T A \bar{x} > 0$. The result is also true if B is a strictly copositive symmetric matrix. For the asymmetric case (A or B or both asymmetric matrices) the foregoing condition is necessary, because by definition,

$$\lambda = \frac{\bar{x}^T A \bar{x}}{\bar{x}^T B \bar{x}} > 0$$

for any solution \bar{x} to the EiCP. A referee remarked that this condition is not sufficient, by providing an example with $n = 2$, $B = I_2$ and

$$A = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}.$$

In fact $x^T A x > 0$ for $x = [1 \ 2]^T$, but the EiCP has no solution $\lambda > 0$, as the $\lambda = -1$ is the unique complementary eigenvalue. It also would be interesting to investigate sufficient weaker conditions guaranteeing that at least one complementary eigenvalue is positive. This is an interesting topic for future research.

3 A Nonlinear Programming Formulation of the Asymmetric EiCP

In the previous section, we have shown that the asymmetric EiCP is equivalent to a finite variational inequality on the simplex. This equivalence has led to a sufficient condition for the existence of a solution to the EiCP. However, the mapping F is not monotone on \mathbb{R}^n and this precludes the EiCP to be solved by the efficient algorithms available for processing VIs [13]. Later in section 6 we include some computational results showing that even PATH is not, in general, able to solve the EiCP. In this paper, we exploit an alternative nonlinear formulation that has been introduced in [22]. Consider again the EiCP and suppose that this problem is solvable. Therefore, there exists $0 \neq \bar{x} \geq 0$ such that

$$\lambda = \frac{\bar{x}^T A \bar{x}}{\bar{x}^T B \bar{x}} > 0.$$

As $\lambda > 0$ in any solution of the EiCP, we can introduce the variable $x_{n+1} = \lambda^{-1}$ and the variables y_i , $i = 1, \dots, n$, defined by $y = x_{n+1}x$ to derive the following equivalent formulation of the EiCP:

$$\begin{aligned} w &= Bx - Ay \\ e^T x &= 1 \\ e^T y - x_{n+1} &= 0 \\ x \geq 0, w \geq 0, y \geq 0, x_{n+1} &\geq 0 \\ x^T w &= 0 \\ \|y - x_{n+1}x\|_2 &= 0, \end{aligned}$$

where $\|\cdot\|_2$ represents the l_2 norm. This prompts the following nonlinear programming problem:

$$\begin{aligned} \text{Minimize} \quad & f(x, y, w, x_{n+1}) = (y - x_{n+1}x)^T(y - x_{n+1}x) + x^T w \\ \text{subject to} \quad & \\ & w = Bx - Ay \\ & e^T x = 1 \\ & e^T y - x_{n+1} = 0 \\ & x \geq 0, w \geq 0, y \geq 0, x_{n+1} \geq 0, \end{aligned}$$

where the EiCP has a solution if and only if this nonlinear program has a global minimum with zero optimal value.

In [22], this nonlinear program has been augmented by $2n$ further constraints to solve the asymmetric EiCP by a branch-and-bound algorithm. To present these constraints, suppose that the EiCP has a solution. Then there must exist positive numbers l and u such that $0 < l \leq x_{n+1} \leq u$ for any global solution (x, y, x_{n+1}, w) of the foregoing nonlinear program. It is obvious that the choice of l and u has an important impact on the complexity of finding a solution for the EiCP. We delay to section 6 the discussion of this issue. Since $y = x_{n+1}x$ in each one of these solutions, then

$$\begin{aligned} lx_i &\leq x_{n+1}x_i = y_i \\ ux_i &\geq x_{n+1}x_i = y_i \end{aligned}$$

for each $i = 1, \dots, n$. We can therefore add these $2n$ conditions and obtain the following nonlinear formulation of the EiCP:

$$\begin{aligned} \text{(NLP)} \quad \text{Minimize} \quad & (y - x_{n+1}x)^T(y - x_{n+1}x) + x^T w \\ \text{subject to} \quad & \\ & w = Bx - Ay \\ & e^T x = 1 \\ & e^T y - x_{n+1} = 0 \\ & lx_i \leq y_i \leq ux_i, \quad i = 1, \dots, n \\ & x \geq 0, w \geq 0, y \geq 0, x_{n+1} \geq 0. \end{aligned}$$

This leads us to the following result.

Theorem 3.1 *The EiCP has a solution \bar{x} , with $\bar{\lambda} = (\bar{x}^T A \bar{x}) / (\bar{x}^T B \bar{x})$ if and only if NLP has a global minimum $(\bar{x}, \bar{y}, \bar{w}, \bar{x}_{n+1})$ with zero optimal value where $\bar{x}_{n+1} = \bar{\lambda}^{-1}$ and $\bar{y} = \bar{x}_{n+1} \bar{x}$.*

Proof: By the construction of Problem NLP, it is sufficient to show that $x_{n+1} > 0$ in any global minimum of this program. But $x_{n+1} = 0$ implies that $e^T y = 0$ and $y = 0$. Since $e^T x = 1$, then there is at least an i such that $x_i > 0$. For such an i we have $lx_i > 0$, which contradicts that $y_i = 0$. Hence, $x_{n+1} > 0$ and the equivalence holds. \square

Thus, we have shown that the EiCP can be equivalently posed as the nonconvex nonlinear program NLP having linear constraints. Due to the nonconvexity of the objective function, a

stationary point of f on the constraint set of the nonlinear program is not necessarily a solution to the EiCP. Next, we establish a necessary and sufficient condition for such a stationary point to be a solution of the EiCP, which is related to the values of the Lagrange multipliers associated with the linear equalities $e^T x = 1$ and $e^T y - x_{n+1} = 0$.

Theorem 3.2 *A stationary point $(\bar{x}, \bar{w}, \bar{y}, \bar{x}_{n+1})$ of Problem NLP is a solution of the EiCP if and only if $\alpha_i = 0, i = 1, 2$, where α_1 and α_2 are the Lagrange multipliers associated with the linear equalities $e^T x = 1$ and $e^T y - x_{n+1} = 0$.*

Proof: Let (x, w, y, x_{n+1}) be a stationary point of the nonlinear program (NLP). Using $w = Bx - Ay$, we can eliminate the variables w_i from NLP to get the following program:

$$\begin{aligned}
& \text{Minimize} && f(x, y, x_{n+1}) = (y - x_{n+1}x)^T(y - x_{n+1}x) + x^T(Bx - Ay) \\
& \text{subject to} && \\
& && Bx - Ay \geq 0 && (r \geq 0) \\
& && e^T x - 1 = 0 && (\alpha_1) \\
& && e^T y - x_{n+1} = 0 && (\alpha_2) \\
& && \left. \begin{array}{l} -lx_i + y_i \geq 0 \\ ux_i - y_i \geq 0 \end{array} \right\} i = 1, \dots, n && \begin{array}{l} (\beta_i \geq 0) \\ (\gamma_i \geq 0) \end{array} \\
& && x \geq 0 && (t \geq 0) \\
& && y \geq 0 && (v \geq 0) \\
& && x_{n+1} \geq 0 && (t_{n+1} \geq 0)
\end{aligned} \tag{3.1}$$

where the dual multipliers are specified within parentheses. Therefore, the stationary point satisfies the following KKT conditions, in addition to (3.1):

$$\begin{aligned}
& \nabla_x f(x, y, x_{n+1}) - B^T r - \alpha_1 e + l\beta - u\gamma = t \\
& \nabla_y f(x, y, x_{n+1}) + A^T r - \alpha_2 e - \beta + \gamma = v \\
& \nabla_{x_{n+1}} f(x, y, x_{n+1}) + \alpha_2 = t_{n+1} \\
& r^T(Bx - Ay) = 0 \\
& \left. \begin{array}{l} \beta_i(-lx_i + y_i) = 0 \\ \gamma_i(ux_i - y_i) = 0 \end{array} \right\} i = 1, \dots, n \\
& t^T x = v^T y = t_{n+1} x_{n+1} = 0 \\
& (r, \beta, \gamma, t, v, t_{n+1}) \geq 0,
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
& \nabla_x f(x, y, x_{n+1}) = -2x_{n+1}(y - x_{n+1}x) + (B + B^T)x - Ay \\
& \nabla_y f(x, y, x_{n+1}) = 2(y - x_{n+1}x) - A^T x \\
& \nabla_{x_{n+1}} f(x, y, x_{n+1}) = -2x^T(y - x_{n+1}x).
\end{aligned} \tag{3.3}$$

Hence, the following equalities hold:

$$-2x_{n+1}(y - x_{n+1}x) + (B + B^T)x - Ay - B^T r - \alpha_1 e + l\beta - u\gamma = t \tag{3.4}$$

$$2(y - x_{n+1}x) - A^T x + A^T r - \alpha_2 e - \beta + \gamma = v \tag{3.5}$$

$$-2x^T(y - x_{n+1}x) + \alpha_2 = t_{n+1}. \tag{3.6}$$

Multiplying both sides of these equalities by x^T , y^T , and x_{n+1} , respectively, and using (3.2), we obtain

$$-2x_{n+1}x^T(y - x_{n+1}x) + x^T(B + B^T)x - x^T Ay - x^T B^T r - \alpha_1 x^T e + lx^T \beta - ux^T \gamma = 0 \quad (3.7)$$

$$2y^T(y - x_{n+1}x) - y^T A^T x + y^T A^T r - \alpha_2 y^T e - y^T \beta + y^T \gamma = 0 \quad (3.8)$$

$$-2x_{n+1}x^T(y - x_{n+1}x) + x_{n+1}\alpha_2 = 0. \quad (3.9)$$

Now, adding the equalities (3.7) and (3.8), and using $x^T B^T r = y^T A^T r$ from (3.2), we have

$$2(y - x_{n+1}x)^T(y - x_{n+1}x) + 2x^T Bx - 2x^T Ay - \alpha_1 x^T e + lx^T \beta - ux^T \gamma - \alpha_2 y^T e - y^T \beta + y^T \gamma = 0.$$

Furthermore, using $lx^T \beta = y^T \beta$, $ux^T \gamma = y^T \gamma$, $e^T x = 1$, and $e^T y = x_{n+1}$ from (3.2), we get

$$2(y - x_{n+1}x)^T(y - x_{n+1}x) + 2x^T(Bx - Ay) = \alpha_1 + \alpha_2 x_{n+1}. \quad (3.10)$$

If $\alpha_1 = \alpha_2 = 0$, then the value of the objective function is zero, which means that the stationary point is a solution of the EiCP. Conversely, suppose that the stationary point (x, w, y, x_{n+1}) of NLP is a solution of the EiCP. Then $f(x, y, w, x_{n+1}) = 0$ and $x_{n+1} > 0$ by the proof of Theorem 3.1. Therefore, (3.9) implies that

$$\alpha_2 = 2x^T(y - x_{n+1}x) = 0.$$

Furthermore, $\alpha_1 = 0$ by (3.10). □

As discussed in [10, 22], if B and A are both symmetric matrices, then the EiCP reduces to finding a stationary point of an appropriate merit function on the simplex Ω . This property is a consequence of the fact that it is possible to prove that the Lagrange multiplier associated with the linear constraint $e^T x = 1$ is exactly zero for this stationary point [30, 22]. In the asymmetric case, however, this result is no longer true, which means that stationary points of the nonlinear program (NLP) may not lead to solutions of the EiCP. Nevertheless, the computation of stationary points for NLP is a valuable tool for the design of an enumerative method to solve with the EiCP. This is discussed in the next section.

4 An Enumerative Algorithm for the EiCP

In this section, we propose a new enumerative algorithm for finding a solution to the EiCP. The algorithm has similarities with an enumerative method for linear complementarity problems discussed in [1, 19] and a branch-and-bound method for the EiCP introduced in [22]. Due to the structure of problem NLP for solving the EiCP, two types of branching strategies are incorporated in the algorithm (similar to the discussion in [22]), namely a branching on the complementary pair of variables and a partitioning of the interval $[l, u]$ for the variable x_{n+1} . These two branching schemes are illustrated in Figure 1.

For a given node k , the restrictions imposed on the branches in the path from this node to the

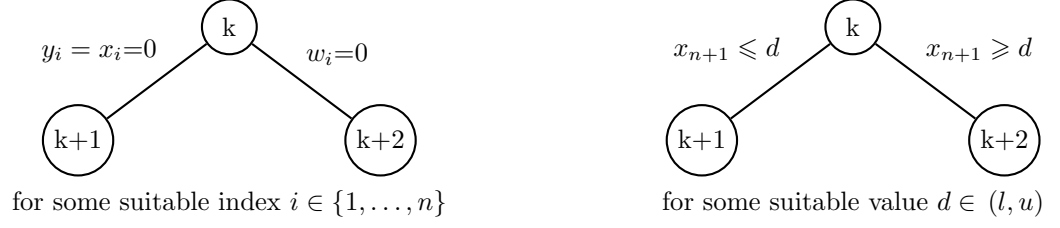


Figure 1: Branching scheme of the algorithm

root node must be considered. Let us assume that these constraints are given by

$$\begin{aligned} \bar{l}x_i &\leq y_i \leq \bar{u}x_i, \forall i \in \bar{J} \\ y_i &= x_i = 0, \forall i \in J \\ w_i &= 0, \forall i \in I, \end{aligned}$$

where $l \leq \bar{l} < \bar{u} \leq u, J \subseteq \{1, \dots, n\}, \bar{J} = \{1, \dots, n\} \setminus J$, and $J \cap I = \emptyset$. Consider the sets

$$K = I \cup J, \bar{I} = \{1, \dots, n\} \setminus I, \text{ and } \bar{K} = \{1, \dots, n\} \setminus K.$$

The subproblem at node k is then given as follows:

$$\begin{aligned} \text{NLP}(k) \quad & \text{Minimize} && f(x, y, w, x_{n+1}) = (y - x_{n+1}x)^T(y - x_{n+1}x) + x_K^T w_{\bar{K}} \\ & \text{subject to} && w = Bx - Ay \\ & && e^T x_{\bar{j}} = 1 \\ & && e^T y_{\bar{j}} - x_{n+1} = 0 \\ & && \bar{l} \leq x_{n+1} \leq \bar{u} \\ & && \bar{l}x_j \leq y_j \leq \bar{u}x_j, j \in \bar{J} \\ & && x_{\bar{j}} \geq 0, w_{\bar{l}} \geq 0, y_{\bar{j}} \geq 0 \\ & && y_j = x_j = 0, j \in J \\ & && w_i = 0, i \in I. \end{aligned} \tag{4.1}$$

At each node k of the tree, the algorithm first searches for a stationary point to the corresponding program $\text{NLP}(k)$. If the objective function value at this stationary point is zero, then a solution to the EiCP is at hand and the algorithm terminates. Otherwise, two new nodes are created by partitioning the node k and the process is repeated. The algorithm also includes heuristic rules for choosing the open node of the list and for deciding which of the two branching strategies of Figure 1 should be used at the selected node k whenever a stationary point having a positive objective value has been found for $\text{NLP}(k)$. The formal steps of the algorithm are presented next.

Enumerative Algorithm

Step 0 Let ϵ be a positive tolerance. Put $k = 1, I = \emptyset, J = \emptyset$, and find a stationary point $(\bar{x}, \bar{y}, \bar{w}, \bar{x}_{n+1})$ of $\text{NLP}(1)$. If $\text{NLP}(1)$ is infeasible, then EiCP has no solution; terminate. Otherwise, let $L = \{1\}$ be the set of open nodes, set $\text{UB}(1) = f(\bar{x}, \bar{y}, \bar{w}, \bar{x}_{n+1})$, and let $N = 1$ be the number of nodes generated.

Step 1 If $L = \emptyset$, terminate; EiCP has no solution. Otherwise, select $k \in L$ such that

$$UB(k) = \min\{UB(i) : i \in L\},$$

and let $(\bar{x}, \bar{y}, \bar{w}, \bar{x}_{n+1})$ be the stationary point found at this node.

Step 2 Let

$$\theta_1 = \max\{\bar{w}_i \bar{x}_i : i \in \bar{K}\} = \bar{w}_r \bar{x}_r$$

$$\theta_2 = \max\{|\bar{y}_i - \bar{x}_{n+1} \bar{x}_i| : i \in \bar{J}\} = |\bar{y}_s - \bar{x}_{n+1} \bar{x}_s|.$$

- (i) If $\max\{\theta_1, \theta_2\} \leq \epsilon$ then $\lambda = \bar{x}_{n+1}^{-1}$ and \bar{x} yield a complementary eigenvalue and eigenvector, respectively (within the tolerance ϵ); terminate.
- (ii) If $\theta_1 > \theta_2$, branch on the complementary variables (\bar{w}_r, \bar{x}_r) associated with θ_1 and generate two nodes, $N + 1$ and $N + 2$.
- (iii) If $\theta_1 \leq \theta_2$, then partition the interval $[\bar{l}, \bar{u}]$ at node k into $[\bar{l}, \tilde{x}_{n+1}]$ and $[\tilde{x}_{n+1}, \bar{u}]$ to generate two nodes, $N + 1$ and $N + 2$, where

$$\tilde{x}_{n+1} = \begin{cases} \bar{x}_{n+1} & \text{if } \min\{(\bar{x}_{n+1} - \bar{l}), (\bar{u} - \bar{x}_{n+1})\} \geq 0.1(\bar{u} - \bar{l}) \\ \frac{\bar{u} + \bar{l}}{2} & \text{otherwise.} \end{cases}$$

Step 3 For each of $t = N + 1$ and $t = N + 2$, find a stationary point $(\tilde{x}, \tilde{y}, \tilde{w}, \tilde{x}_{n+1})$ of $NLP(t)$. If $NLP(t)$ is feasible, set $L = L \cup \{t\}$ and $UB(t) = f(\tilde{x}, \tilde{y}, \tilde{w}, \tilde{x}_{n+1})$. Set $L = L \setminus \{k\}$ and return to Step 1.

The following result establishes that the enumerative algorithm is able to find a solution to a solvable EiCP.

Theorem 4.1 *Either the algorithm terminates finitely with a solution to the EiCP including possibly an indication that no such solution exists or an infinite binary tree is created such that along any infinite branch of this tree, any accumulation point of stationary point of $NLP(k)$ generated by the algorithm solves the EiCP.*

Proof: The case of finite convergence is obvious. Otherwise, suppose that an infinite tree is generated. An accumulation point of the stationary points of $NLP(k)$ along an infinite branch for a suitable index set \mathcal{K} yields in the limit

$$\{\gamma^k\}_{\mathcal{K}} \longrightarrow \gamma^*, \quad [l_k, u_k]_{\mathcal{K}} \longrightarrow [l^*, u^*],$$

where $\gamma^k = (w^k, x^k, y^k, x_{n+1}^k)$ is the solution at node k , and $\{l_k, u_k\}$ are the lower and upper bounds for the variable x_{n+1} at node k . Following exactly the same arguments used in the proof of Theorem 4 of [22], we can show that

$$x_{n+1}^* = l^* \text{ or } x_{n+1}^* = u^*$$

with $\theta_1 \rightarrow 0$ and $\theta_2 \rightarrow 0$, and

$$e^T y^* = x_{n+1}^* \quad (4.2)$$

$$y^* = x_{n+1}^* x^* \quad (4.3)$$

$$w^* = -x_{n+1}^* A x^* + B x^* \quad (4.4)$$

$$w_i^* x_i^* = 0, \quad i = 1, \dots, n. \quad (4.5)$$

By the definition of the algorithm, we have $x^* \geq 0$, $w^* \geq 0$, $y^* \geq 0$, and $0 < l \leq l^* \leq x_{n+1}^* \leq u^* \leq u$. Hence, $\gamma^* = (w^*, x^*, y^*, x_{n+1}^*)$ is a feasible solution of NLP. Furthermore, by (4.3) and (4.5), the value of the objective function at γ^* is zero. Thus, x^* and $\lambda^* = \frac{1}{x_{n+1}^*}$ solve the EiCP. \square

Note that there is no restriction on the classes of the matrices A and B for Theorem 4.1 to hold.

It follows from this theorem that the enumerative algorithm is always able to find a solution for the EiCP whenever it exists. If the EiCP has no solution, then the algorithm fathoms nodes finitely to terminate with an empty list of open nodes, where $UB(r) > 0$ for all visited nodes r such that $NLP(r)$ is feasible. This also means that NLP has a global minimum with a positive objective function value.

An important issue for the algorithm is the choice of the initial values l and u utilized in the program NLP. The value l should be near-zero and u should be large enough to guarantee the existence of an eigenvalue \bar{x}_{n+1} inside this interval. However, large values of u may result in computational difficulties for the algorithm. A heuristic procedure for the choice of these values l and u is discussed in [22] and has been used in the present enumerative method. As such, our algorithm essentially seeks a solution to the EiCP that has an eigenvalue within the specified interval $[l, u]$, if it exists.

5 Computation of the Minimum and Maximum Complementary Eigenvalues

In practice, it is of interest to compute the minimum and maximum eigenvalues for the EiCP within a given interval $[l, u]$ with $l > 0$ and $u > l$. We describe next a simple technique to find such extremal eigenvalues for the EiCP based on the enumerative algorithm introduced in the previous section. Suppose that we wish to compute the minimum eigenvalue $\lambda > 0$ of the EiCP. Since $x_{n+1} = \lambda^{-1}$, then the maximum value of x_{n+1} in the interval $[l, u]$ should be computed. The enumerative algorithm discussed above can be employed initially to find an EiCP solution with $\bar{x}_{n+1} \in [l, u]$. Now, either $\bar{x}_{n+1} = u$ and the minimum eigenvalue is at hand or a smaller eigenvalue may exist corresponding to $x_{n+1} \in [\bar{x}_{n+1}(1 + \epsilon^+), u]$ for a small $\epsilon^+ > 0$. The enumerative algorithm can be applied again for this new interval $[l, u]$, where $l \equiv \bar{x}_{n+1}(1 + \epsilon^+)$. If the resulting EiCP has no solution, then $\lambda = \bar{x}_{n+1}^{-1}$ is an ϵ^+ -approximate minimum complementary eigenvalue. Otherwise, the EiCP is solvable in the new interval and a revised value \tilde{x}_{n+1} is computed at an EiCP solution such that $\bar{x}_{n+1} < \tilde{x}_{n+1}$. The procedure is then repeated. The steps of this sequential algorithm are presented below.

Sequential Enumerative Method

- Step 0** Let $\epsilon^+ > 0$ be a positive tolerance. Find a solution (\bar{x}, \bar{x}_{n+1}) of the EiCP by the enumerative algorithm. If such a solution does not exist, terminate the algorithm with an indication that the EiCP has no solution with $x_{n+1} \in [l, u]$.
- Step 1** Let $l = \bar{x}_{n+1}(1 + \epsilon^+)$ and apply the enumerative algorithm to solve the EiCP in the new interval $[l, u]$ via the corresponding NLP.
- Step 2** If NLP is infeasible, or if the enumerative algorithm terminates (finitely) with an indication that no EiCP solution exists, then $\lambda = \bar{x}_{n+1}^{-1}$ is the smallest eigenvalue and \bar{x} is the corresponding eigenvector. Otherwise, let $(\tilde{x}, \tilde{y}, \tilde{w}, \tilde{x}_{n+1})$ be the new EiCP solution found. Replace $\bar{x} \leftarrow \tilde{x}$, and $\bar{x}_{n+1} \leftarrow \tilde{x}_{n+1}$, and go to Step 1.

The algorithm for finding the maximum eigenvalue λ (smallest x_{n+1} in a given interval $[l, u]$) is similar to the procedure above. However, it is the upper-bound u that has to be updated, which can be done by resetting $u = \bar{x}_{n+1}(1 - \epsilon^+)$. In either case, the procedure loops finitely because of the discrete steps taken, requiring $O(\log(u/l)/\log(1 + \epsilon^+))$ and $O(\log(l/u)/\log(1 - \epsilon^+))$ sequential iterations, respectively, for the two cases.

It immediately follows from the description of the algorithm, that the last NLP to be solved by the enumerative algorithm has a global minimum with a positive objective function value. Contrary to the case of a solvable EiCP, given an interval for x_{n+1} , it is important to generate positive lower bounds at open nodes for an unsolvable EiCP to accelerate the convergence process by virtue of pruning the tree at such nodes. Lower bounds for the enumerative method have been designed in [22]. We recommend a switching point within the enumerative process where the algorithm computes lower bounds instead of upper bounds. Heuristic rules for this switching point are presented below.

Rules for switching from the computation of upper bounds to lower bounds

- (i) Switch at a node k and for all its descendants if

$$\bar{u} - \bar{l} \leq p(u - l),$$

where $0 < p < 1$ is a specified threshold, $[l, u]$ is the original interval for the variable x_{n+1} , and $[\bar{l}, \bar{u}]$ is the current interval for x_{n+1} at node k .

- (ii) Switch at a node k and for all its descendants if

$$|I \cup J| \geq qn,$$

where $0 < q < 1$ is a specified threshold and I and J represent the set of fixed \bar{w}_i and \bar{x}_i variables at node k .

The choice of p and q depends on the number of complementary variables and on the amplitude of the interval $[l, u]$ of the EiCP. In the next section, some experience is reported with the sequential algorithm incorporating this switching procedure.

6 Computational Experience

In this section, we report computational experience for the proposed algorithms and implementation strategies. All the experiments have been performed on a Pentium III 1.0 GHz machine with 2 gigabytes of RAM memory. The enumerative method was coded in GAMS [4] and the solutions of $NLP(k)$ associated with the different nodes k were obtained by the MINOS [28] solver. The branch-and-bound algorithm [22] was implemented in FORTRAN 77 with version 10 of the Intel FORTRAN compiler [6]. The solver BARON [31, 35], with a GAMS implementation, was also used to solve the NLP. The BARON code implements a branch-and-reduce algorithm for global optimization.

For our test EiCP problems, B was taken as a real positive definite (PD) matrix and A as a real matrix belonging to the **pentadiagonal** [29, page 380] or **Fathy** [29, page 311] classes, or taken from the **Matrix Market** repository [27], or in some cases as a randomly generated matrix. For each set of problems, the initial interval bounds for the variable x_{n+1} , l and u , have been chosen by trial and error so that EiCP has a solution $x_{n+1} \in [l, u]$.

The test problems of Table 1 are scaled according to the described procedure in [22, Section 5]. Scaling is an important tool for improving the efficacy of the algorithm, particularly when A is ill-conditioned. In our experiments, we have chosen symmetric matrices that have also been used in tests with other alternative techniques [21] as well as asymmetric matrices. We have also solved the problems discussed in [8] by modifying the enumerative algorithm in order to process the corresponding MEiCPs.

6.1 Symmetric EiCPs

The test cases for this experiment are from [21], where B is the identity matrix (I_n) and A is a matrix from the **pentadiagonal** or **Fathy** classes. These matrices are symmetric positive definite, but ill-conditioned. In Table 1 we present the results with the enumerative method, where x_{n+1} is the computed inverse of the complementary eigenvalue found. The value of the tolerance ϵ has been set equal to 10^{-6} in these tests, which is sufficiently stringent for a global optimization algorithm. Furthermore, l and u have been set equal to 10^{-2} and 500, respectively. We have used $l = 10^{-3}$ for the **Fathy** class with dimensions 300, 400, and 500, because the corresponding inverse eigenvalue is quite small for these problems. The notations N_C , N_P , and N stand for the number of branchings performed on the complementary variables, the number of partitions of the interval for x_{n+1} , and the total number of investigated nodes required by the algorithm. Furthermore, ‘Order’ represents the dimension (n) of the EiCP, It is the number of iterations performed (pivotal operations), and T is the CPU time in seconds.

The results show that the enumerative method solves all the test problems at the initial node itself, without performing any branching.

As discussed in [21, 30], symmetric EiCPs can be solved by finding a stationary point of an appropriate merit function on the simplex Ω . A computational study of the efficiency of active-set, interior-point, and spectral projected-gradient (SPG) algorithms for solving the symmetric EiCP is reported in [21]. The results displayed in Table 1 and Table 4 of [21] show that, for the same termination tolerance, the enumerative method outperforms the active-set and interior-point algorithms, which is quite remarkable. The reasons for this behavior have to do with the sparsity

A	Order (n)	x_{n+1}	T	It	N_P	N_C	N
Fathy	100	2.4489E-2	0.11	328	0	0	1
	200	1.2226E-2	0.56	599	0	0	1
	300	7.6815E-3	1.83	842	0	0	1
	400	5.3621E-3	3.64	894	0	0	1
	500	4.2104E-3	7.06	946	0	0	1
pentadiagonal	100	1.0000	0.14	640	0	0	1
	200	1.0000	0.57	1330	0	0	1
	300	1.0000	1.40	2130	0	0	1
	400	1.0000	2.63	2956	0	0	1
	500	1.0000	4.48	3956	0	0	1

Table 1: Performance of the enumerative algorithm for symmetric EiCPs.

of the constraints and of the Hessian of the objective function of NLP and from the fact that no branching is required for processing these problems. However, the SPG algorithm is still more efficient than the enumerative method for this special symmetric case.

We have also tested the solution of these EiCPs by exploiting its reduction to a VI discussed in section 2. The Mixed Nonlinear Complementarity Problem (MNCP) equivalent to this VI is given by

$$\begin{aligned}
 F(x) - \lambda e &= w \\
 e^T x &= 1 \\
 x &\geq 0, w \geq 0 \\
 \lambda &\text{ unrestricted}
 \end{aligned}$$

We have applied PATH for finding a solution of the corresponding MNCPs associated with the symmetric EiCPs introduced before. For all the test problems, PATH was unable to find a solution to these EiCPs. This clearly shows that PATH is, in general, unable to solve symmetric or asymmetric EiCPs.

6.2 Asymmetric EiCPs

The first set of asymmetric test problems were taken from [22], where $B = I_n$ and A is an asymmetric matrix whose elements were randomly generated and uniformly distributed in the intervals $[-1, 1]$ or $[0, 1]$. The letters “ma” and “mp” are used to identify these intervals, respectively, in Table 2. The number presented after these letters represents the order (n) of the matrices A and B . Furthermore, the interval $[l, u]$ of the NLP associated with the EiCP is also specified in Table 2. The results reported in Table 2 include the performances of the branch-and-bound algorithm discussed in [22], the proposed enumerative method, and the BARON/GAMS procedure for solving these EiCPs, which are known to have solutions. To allow a comparison of results, we used the termination tolerance of 10^{-6} for all these methods. The symbol (***) used in this table, as well

A	[l, u]	Branch-and-bound [22]							Enumerative method							BARON/GAMS		
		x_{n+1}	T	It	Np	Nc	N	x_{n+1}	T	It	Np	Nc	N	x_{n+1}	T	N		
ma6	[0,3]	1.1056	0.10	302	31	22	93	1.1057	0.07	128	2	0	5	1.1057	0.03	1		
ma10	[0,3]	2.0736	0.54	4386	51	152	357	2.0736	0.60	275	13	10	46	2.0736	1.63	19		
ma20	[0,1]	0.5580	0.96	9333	42	107	276	0.5580	0.05	288	1	0	3	0.5580	0.03	1		
ma30	[0,3]	0.8740	15.75	108944	190	1149	2516	0.6610	0.32	1350	14	0	26	1.1387	16.36	10		
ma40	[0,1]	0.3214	12.16	70148	28	266	447	0.3213	0.07	460	0	0	1	0.3213	0.13	1		
ma50	[0,1]			***				0.4121	0.06	343	0	0	1		***			
mp6	[0,1]	0.3716	0.03	118	11	3	22	0.3716	0.01	18	0	0	1	0.3716	0.01	1		
mp10	[0,1]	0.1919	0.03	180	7	2	13	0.1919	0.01	31	0	0	1	0.1919	0.01	1		
mp20	[0,1]	0.1014	0.10	464	20	1	37	0.1014	0.01	54	0	0	1	0.1014	0.02	1		
mp30	[0,1]	6.655E-2	0.16	753	17	2	34	6.655E-2	0.01	80	0	0	1	6.655E-2	0.04	1		
mp40	[0,1]	5.005E-2	0.26	970	12	2	22	5.005E-2	0.02	104	0	0	1	5.005E-2	0.08	1		
mp50	[0,1]	3.936E-2	0.59	1717	19	1	36	3.936E-2	0.02	151	0	0	1	3.936E-2	0.15	1		

Table 2: Performance of enumerative, branch-and-bound, and BARON algorithms for asymmetric FiCPs.

as in the subsequent tables, indicates that the particular algorithm was unable to solve a given problem within the set limits.

The results show that the enumerative method is more efficient than the other two algorithms for processing solvable EiCPs, as it requires a small number of branchings, fewer iterations, and lesser CPU time to terminate. Furthermore, the gap between the performances of the enumerative and the branch-and-bound algorithms increases with the dimension of the processed EiCP. Note that the branch-and-bound method of [22] and BARON were unable to find a solution for one of the EiCPs. A possible reason for this better behavior of the enumerative method over the two alternative branch-and-bound algorithms lies on the fact that solution of the EiCP is a global minimum of the NLP formulation discussed in section 3 with a zero optimal value. The knowledge of this global optimal value is a stopping criterion for the termination of the enumerative algorithm. On the contrary, branch-and-bound algorithms are based on the construction of good lower-bounds and do not exploit this condition as a stopping criterion.

A	Order (n)	Enumerative Method						BARON/GAMS		
		x_{n+1}	T	It	N_P	N_C	N	x_{n+1}	T	N
bfw62a	62	1.3139	0.01	4	0	0	1	0.3883	0.19	1
ck104	104	1.5654	0.01	14	0	0	1	3.8944	7.75	1
fs_183_1	183	1.0000	0.01	4	0	0	1	383.5406	5.15	1
gre_115	115	7.8750	0.01	10	0	0	1	7.8750	3.84	1
impcol_c	137	7.6771	0.36	959	0	0	1	0.1000	4.46	1
olm100	100	1.624E-2	0.01	3	0	0	1	***		
rw136	136	30.9687	0.02	40	0	0	1	30.9687	10.18	1
west0167	167	0.6793	0.04	99	0	0	1	0.3607	26.06	1

Table 3: Performance of the enumerative algorithm and BARON for asymmetric EiCPs with matrices from the Matrix Market repository.

In a second set of test problems, B was taken as a matrix from the **Murty** class and A as an asymmetric indefinite matrix from the **Matrix Market** repository. The results for the proposed enumerative method and BARON/GAMS using these problems are presented in Table 3. Note that the dimensions of these EiCPs are larger than the previous ones, but the conclusion about the comparative performances of the algorithms is similar.

As discussed before, the values of l and u have to be chosen a priori so that at least an eigenvalue lies inside the interval $[l, u]$. To undergo a sensitivity analysis on the values of l and u , we have fixed $l = 0.1$ and change u from a relatively small value to a very large one. The results of this sensitivity analysis on an asymmetric EiCP with the matrix **ma30** and a symmetric EiCP with the matrix **pentadiagonal** of order 100 are displayed in Tables 4 and 5 and clearly show that the efficiency of the algorithm does not seem to be quite influenced by the magnitude of the interval $[l, u]$.

u	x_{n+1}	T	It	N_P	N_C	N
10	0.5769	0.03	170	0	0	1
100	1.8299	0.26	2099	7	0	14
10^4	0.5769	0.03	302	0	0	1
10^6	0.5769	0.04	313	0	0	1

Table 4: Solution of EiCP with an asymmetric matrix and four different intervals for the variable x_{n+1} .

u	x_{n+1}	T	It	N_P	N_C	N
10	0.7702	0.41	1644	0	0	1
100	0.7539	0.69	2886	0	0	1
10^4	0.9875	0.96	4167	0	0	1
10^6	1.0000	0.87	3596	0	0	1

Table 5: Solution of EiCP with a symmetric matrix and four different intervals for variable x_{n+1} .

6.3 Engineering Application

The results for using the enumerative method to solve three EiCPs associated with the engineering model described in [8] are presented in Table 6. In this table, λ is the computed complementary eigenvalue of the MEiCP, and $|s|$ is the number of nodes in impending slip [8] and equals the number of complementary variables of the MEiCP. The values of l and u have been taken as 10^{-2} and 500, respectively. As the value of x_{n+1} is usually quite small for the MEiCP- λ^2 , we have used $l = 10^{-6}$ for these three problems.

	Problem	Order (n)	$ s $	λ	T	It	N_P	N_C	N
MEiCP- λ	I	9	3	4.0052	0.01	64	0	0	1
	II	9	3	3.2157	0.01	71	0	0	1
	III	35	5	3.1866	0.01	157	0	0	1
MEiCP- λ^2	I	9	3	3.871E-3	0.01	24	0	0	1
	II	9	3	2.199E-4	0.01	15	0	0	1
	III	35	5	2.756E-5	0.04	540	3	0	5

Table 6: Performances of the enumerative algorithm for the engineering problems discussed in [8].

The three examples are associated with models of rectangular blocks sliding on a surface that may be considered as a rigid obstacle. Problem I models a square block and the other two problems model a rectangular block where the height is half the length. In fact, the second and third problems represent the same problem with different meshes. Also, we modified the code for the enumerative method to deal with unrestricted variables. The results of Table 6 show that the enumerative method was able to solve these MEiCPs efficiently without branching on the

complementary variables.

6.4 Finding Maximum and Minimum Eigenvalues

Finally, Table 7 presents the results for the sequential algorithm for computing the maximum eigenvalue for several test cases of dimension up to 30. Here, N_{sol} represents the number of computed EiCP solutions. The matrix A is of the **Fathy** class, or **pentadiagonal**, or **tridiagonal** [16] with nonzero elements given by

$$\left. \begin{aligned} A(i, i) &= 4, \quad i = 1, \dots, n \\ A(i-1, i) &= -1 \\ A(i, i-1) &= -1 \end{aligned} \right\} i = 2, \dots, n.$$

The value $\epsilon^+ = 0.05$ has been used in the updating formula in Step 2 of the sequential enumerative algorithm. Furthermore, † represents the algorithm has not been able to confirm that the maximum eigenvalue was found after 2500 nodes.

As discussed in Section 5, for the sequential enumerative algorithm, we have utilized some heuristic rules for switching from the computation of upper bounds to lower bounds. These rules are based on two threshold parameters, p and q , that govern the switching nodes. Based on some preliminary tests, we have used $p = q = 1/2$ for these runs.

A	n	λ	N_{sol}	It	N_P	N_C	N	T
Fathy	5	2.3449	1	196	0	5	12	0.07
	10	4.3634	1	946	0	10	22	0.17
	15	6.3875	1	2122	0	15	32	0.35
	20	8.4124	1	3570	0	20	42	0.68
	25	10.4386	1	5822	0	25	52	1.28
	30	12.4646	1	8632	0	30	62	2.14
pentadiagonal	5	1.2357	2	268	0	7	17	0.10
	10	1.2887	3	4974	15	90	212	0.92
	15	1.3080	3	28833	8	254	528	4.45
	20	1.3045	3	87108	40	721	1525	11.30
	25	1.2954	3	150381	152	1101	2509	16.70 [†]
	30	1.3092	3	162838	100	1151	2505	17.30 [†]
tridiagonal	5	4.0000	1	93	0	12	26	0.02
	10	4.0000	1	1318	0	143	288	0.20
	15	4.0000	1	15347	0	1251	2503	2.44 [†]
	20	4.0000	1	33786	0	1251	2503	4.21 [†]
	25	4.0000	1	42279	0	1251	2503	5.27 [†]
	30	4.0000	1	47048	0	1251	2503	6.15 [†]

Table 7: Performance of the sequential enumerative algorithm for finding the maximum complementary eigenvalue of symmetric EiCPs.

The numerical results show that the algorithm consumes a reasonable amount of effort for solving such problems. Furthermore, the number of updates (N_{sol}) for x_{n+1} is quite small. Actually, $N_{sol} = 1$ in many cases, which means that the first complementary eigenvalue computed by the enumerative method is the maximum complementary eigenvalue of the EiCP. The algorithm is usually quite fast to obtain the maximum eigenvalue, but confirming that such a maximum has been found is usually quite time consuming. In some examples, the algorithm attained the maximum number of nodes allowed without confirming that the maximum eigenvalue has been achieved.

In the next experience, we have tried to find the maximum and the minimum eigenvalues for some of the EiCPs with asymmetric matrices introduced in section 6.2. We set the tolerances $\epsilon^+ = 0.05$ and $\epsilon^- = 0.05$ as before, and we used $l = 10^{-2}$ and $u = 500$ in all the experiences. As before, † represents that the algorithm was unable to confirm that the maximum eigenvalue was found after 2500 nodes.

A	λ	N_{sol}	It	N_P	N_C	N	T
ma6	1.1091	3	1708	5	53	120	0.61
ma10	0.4823	1	27334	66	779	1691	5.67
ma20	1.7921	2	106733	17	1235	2505	11.90†
ma30	1.7334	1	136850	13	1238	2503	15.70†
ma40	3.1125	1	192706	15	1236	2503	30.40†
ma50	2.7017	3	287740	16	1240	2514	50.40†

Table 8: Performance of the sequential enumerative algorithm for finding the maximum eigenvalue of asymmetric EiCPs.

A	λ	N_{sol}	It	N_P	N_C	N	T
ma6	0.9044	1	98	13	4	36	0.01
ma10	0.4823	1	8804	39	275	629	1.35
ma20	1.5845	1	808	14	1	31	0.08
ma30	0.1819	6	79773	84	1182	2535	8.99†
ma40	2.8609	2	22154	23	208	465	2.86
ma50	0.5443	10	129771	148	1164	2629	23.60†

Table 9: Performance of the sequential enumerative algorithm for finding the minimum eigenvalue of asymmetric EiCPs.

The numerical results displayed in Tables 7, 8 and 9 seem to indicate that the sequential enumerative algorithm has a similar performance for solving asymmetric and symmetric EiCPs.

7 Conclusions

In this paper, an asymmetric Eigenvalue Complementary Problem (EiCP) is discussed. A sufficient condition for the existence of a solution was first established. An enumerative algorithm was

proposed for finding a solution of the EiCP, if it exists. The algorithm adopts a new nonlinear programming formulation (NLP) of the EiCP motivated by, but different from, that introduced in [22] and implements branching strategies similar to those incorporated in the branch-and-bound method discussed in [22]. The proposed algorithm, however, has an added facility to possibly terminate early (even at the root node in some cases) based on a necessary and sufficient optimality utilized in this paper. A sequential algorithm based on these enumerative and branch-and-bound algorithms was also proposed to compute the maximum and the minimum complementary eigenvalues. Computational experience reported on a variety of problems demonstrate the relative efficiency of the new proposed methodologies over the algorithm of [22] as well as over the commercial software BARON. We believe that this type of enumerative algorithm can be useful for the solution of other structured complementary problems that often appear in engineering applications. This is a topic recommended for future investigation.

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