On the use of bilevel programming for solving a structural optimization problem with discrete variables

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Summary. In this paper, a bilevel formulation of a structural optimization problem with discrete variables is investigated. The bilevel programming problem is transformed into a Mathematical Program with Equilibrium (or Complementarity) Constraints (MPEC) by exploiting the Karush-Kuhn-Tucker conditions of the follower's problem.

A complementarity active-set algorithm for finding a stationary point of the corresponding MPEC and a sequential complementarity algorithm for computing a global minimum for the MPEC are analyzed. Numerical results with a number of structural problems indicate that the active-set method provides in general a structure that is quite close to the optimal one in a small amount of effort. Furthermore the sequential complementarity method is able to find optimal structures in all the instances and compares favorably with a commercial integer program code for the same purpose.

Key Words: Structural optimization, mixed integer programming, global optimization, complementarity.

1 Introduction

In the last few decades, Structural Optimization has become an area of increasing interest and intense research [1, 3, 5, 10, 9, 12, 20, 22, 23, 25]. These models are formulated as challenging optimization problems representing the elastoplastic laws of mechanics and searching for a structure with the least

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volume. A quite general structural optimization model has been introduced in [8] whose formulation leads into a bilinear program with linear and bilinear constraints. The variables of this optimization problem are associated to the coordinates at each node of the structure and the cross sectional areas. The latter should belong to a fixed set of admissible values. Furthermore each feasible solution is characterized by a vector x, whose components are 1 or 0, depending on the corresponding bar to be or not to be included in the optimal structure.

As discussed in [8], this bilinear program with discrete variables can be reduced into a mixed integer zero-one linear program. Computational experience reported in [8] shows that the model is quite appropriate for finding a structure that requires small amount of material. A commercial code, such as OSL [18], can in general find an optimal solution for the optimization problem when the number of nodes and pre-fixed values for the cross-sectional areas are small. However, the algorithm faces difficulties in finding such a solution when the dimension of the problem increases.

A mixed integer zero-one linear program can be shown to be equivalent to a Linear Bilevel Programming Problem [2]. By exploiting the Karush-Kuhn-Tucker conditions of the follower's problem it is possible to reduce this bilevel program into a Mathematical Programming Problem with Equilibrium (or Complementarity) Constraints of the following form

$$\begin{array}{lll} \text{MPEC:} & \textit{Minimize} & c^T z + d^T y \\ & \textit{subject to} & Ew = q + Mz + Ny \\ & & z \geq 0, \quad w \geq 0 \\ & & y \in K_y \\ & & z^T w = 0 \end{array} \tag{1}$$

where $q \in \mathbb{R}^p$, c, $z \in \mathbb{R}^n$, d, $y \in \mathbb{R}^m$, M, $E \in \mathbb{R}^{p\times n}$, $N \in \mathbb{R}^{p\times m}$ and

$$K_y = \{y \in \mathbb{R}^m : \ Cy = b, \ y \geq 0\}$$

with $C \in \mathbb{R}^{l\times m}$ and $b \in \mathbb{R}^l$.

Due to its structure, an active-set methodology seems to be quite appropriate to process this MPEC. A complementarity active-set (CASET) algorithm has been introduced in [16] to find a stationary point for the MPEC. The procedure maintains complementarity during the entire process and has been shown to converge to a stationary point under reasonable hypotheses. Computational experience reported in [16] has shown that the proposed algorithm is in general quite efficient to process moderate and even large MPECs.

A Sequential Linear Complementarity (SLCP) algorithm has been introduced in [14] to find a global minimum for a linear MPEC. The algorithm finds a sequence of stationary points of the MPEC with strictly decreasing value. The last stationary point of this sequence is shown to be a global minimum of the MPEC. Computational experience reported in [13, 14, 15] indicates that the algorithm is quite efficient to find a stationary point that is a global minimum of the MPEC, but faces difficulties in establishing that such a global minimum has been achieved.

In practice, engineers search for a structure that serves their purposes, that is, a feasible solution of the mixed integer program with a small objective function value is requested. As each stationary point of the MPEC corresponds to a feasible solution of its equivalent zero-one integer program, then both the CASET and SLCP algorithms seem to be valid approaches to find a good structure for the structural model. In this paper we investigate how these two algorithms perform for a number of structures presented in [8]. The experiments indicate that the CASET algorithm is able to find in general a good structure in a small amount of effort. On the other hand, the SLCP algorithm has always found a global optimal structure for the model. Furthermore the computational effort required by the SLCP algorithm tends to become much smaller than the one needed by an integer program code as the dimension of this problem increases.

The organization of the paper is as follows. In Section 2 the structural model and its formulation are introduced. Section 3 is devoted to the equivalence between a zero-one mixed integer program and an MPEC. The algorithms CASET and SLCP are briefly described in sections 4 and 5. Finally computational experience with these algorithms on a set of structural problems and some conclusions are included in the last two sections.

2 A topological optimization model

The admissible structural domain is referenced by a bidimensional cartesian system Oxy, in which the various alternative solutions for the problem under consideration can be developed. A discretisation [26] of this domain is then considered in which the mesh is composed by bar elements joined at the nodal points.

The structural domain is submitted to the various actions defined in the safety code [6] such as the structural self-weight, wind, earthquake and so on. These actions lead to different l loading conditions, each of them is represented by nodal point loads

$$f^l = \begin{bmatrix} f_x^l \\ f_y^l \end{bmatrix}.$$

Some of these loads are reactions r^l, when the associated nodes are connected to the exterior. The nodal displacements

$$u^l = \begin{bmatrix} u_x^l \\ u_y^l \end{bmatrix}$$

are associated to these nodal forces. The stress field within each bar element i for loading condition l can be determined from its axial load e_i^l, while the strain field is given by the axial deformation d_i^l.

The fundamental conditions to be satisfied in the serviceability limit states are equilibrium, compatibility, boundary conditions and elastic constitutive relations of the structural material.

Equilibrium has to be verified at a nodal level and relates the elastic axial bar forces e^l_e with support reactions r^l_e and applied nodal loads f^l by

$$C^T e^l_e - B r^l_e - f^l = 0, \tag{2}$$

where C and B are matrices depending on the structural topology.

The compatibility conditions imply equal displacement for all the bar ends joining at the same node and can be expressed as

$$d^l_e = C u^l, \tag{3}$$

where d^l_e is the bar deformation vector, u^l is the nodal displacement vector and C is the connectivity matrix already used in (2).

The forces e^l_e in the structural bars are related to the bar deformations d^l_e by linear elastic constitutive relations given by the so-called Hooke's Law

$$e^l_e = K D_A d^l_e, \tag{4}$$

where $D_A = diag\{A_i\}$, with A_i a discrete variable associated to the cross-sectional area of bar i and $K = diag\{E_i h_i^{-1}\}$, with $E_i > 0$ the Young's modulus of bar i and h_i its length . It follows from (2), (3) and (4) that

$$C^T K D_A C u^l - B r^l_e - f^l = 0. \tag{5}$$

The structural boundary conditions are given by

$$u^l_m = 0 \tag{6}$$

for the nodes m connected to supports with zero displacement.

The nodal displacements should comply with the upper and lower bounds defined in the safety codes

$$u_{min} \leq u^l \leq u_{max}. \tag{7}$$

The ultimate limit states can be considered on the basis of the Plasticity Theory. According to the Static Theorem, the fundamental conditions to be fulfilled are equilibrium, plasticity conditions and boundary conditions.

The equilibrium conditions are given in a similar form to (2) by

$$C^T e^l_p - B r^l_p - \lambda f^l = 0, \tag{8}$$

where e^l_p is the plastic force vector, r^l_p the plastic reaction vector and λ is a partial safety majoration factor for the nodal forces corresponding to the applied actions, prescribed in structural safety codes [6, 7].

The plasticity conditions can be expressed as

$$e_{min} \leq e_p^l \leq e_{max}, \tag{9}$$

where e_{min} and e_{max} are the minimum and maximum admissible values for the element forces defined in the code [7].

The conditions (5), (6), (7), (8) and (9) considered so far are satisfied by many solutions in which some bars have zero force. A vector x is further introduced in the model such that each variable x_i is associated with bar i and takes value 1 or 0, depending on the bar i to be or not to be included in the solution.

The force in a generic bar i can then be replaced by the product $x_i e_{p_i}^l$ yielding a null force in non-existing bars. So the axial bar force must verify the following conditions

$$D_x e_{min} \leq e_p^l \leq D_x e_{max}, \tag{10}$$

where

$$D_x = diag(x_i). \tag{11}$$

Furthermore the diagonal matrix D_A takes the form $D_A D_x$. The model seeks an optimal solution corresponding to the minimum use of structural material V. If A_i is the cross-sectional area of bar i and h_i is its length, then the objective function takes the form

$$V = \sum_i x_i A_i h_i. \tag{12}$$

The optimization problem described by the equations (2-11) consists of minimizing a bilinear function in variables x_i and A_j on a set of linear and bilinear constraints. Furthermore x_i are zero-one variables and the variables A_i can only assume values in a discrete set of positive fixed numbers A_{ik}, $k = 1, \ldots, N_i$. These variables can be transformed into a set of zero-one variables y_{ik} by using traditional manipulations, as described in [8]. On the other hand, bilinear terms such as $x_i y_{ik}$ can be transformed into variables by exploiting the so-called Reformulation-Linearization Technique RLT [8, 24]. These transformations lead into a zero-one mixed-integer program, as shown below.

Unfortunately optimal structures associated to the optimization problem may be not kinematically stable. In order to avoid such type of structures the so-called Grubler's Criterion [11] is exploited. As discussed in [8], this criterion can be analytically presented by some further linear constraints.

All these considerations lead into the following formulation of the structural model [8] under study in this paper.

OPT: $$Minimize\ V = \sum_{i=1}^{nb} \left(\sum_{k=1}^{N_i} A_{ik} y_{ik} \right) h_i$$

subject to

$$\sum_{i=1}^{nb} M_{ji} \left(\sum_{k=1}^{N_i} A_{ik} q_{ik}^l \right) - \sum_{m=1}^{na} B_{jm} r_{e_m}^l - f_j^l = 0 \quad (13)$$

$$d^l = Cu^l \quad (14)$$

$$u_{min} \leq u^l \leq u_{max} \quad (15)$$

$$d_{min_i} y_{ik} \leq q_{ik}^l \leq d_{max_i} y_{ik} \quad (16)$$

$$d_{min_i} \left(1 - \sum_{k=1}^{N_i} y_{ik} \right) \leq d_i^l - \sum_{k=1}^{N_i} q_{ik}^l \quad (17)$$

$$d_i^l - \sum_{k=1}^{N_i} q_{ik}^l \leq d_{max_i} \left(1 - \sum_{k=1}^{N_i} y_{ik} \right) \quad (18)$$

$$u_{j_m}^l = 0 \quad (19)$$

$$-C^T e_p^l + B r_p^l + \lambda f^l = 0 \quad (20)$$

$$t_{min_i} \sum_{k=1}^{N_i} A_{ik} y_{ik} \leq e_{p_i}^l \leq t_{max_i} \sum_{k=1}^{N_i} A_{ik} y_{ik} \quad (21)$$

$$z_n \leq \sum_{i \in I(n)} \sum_{k=1}^{N_i} y_{ik} \leq |I(n)|\, z_n \quad (22)$$

$$2 * \sum_{n=1}^{nn} z_n - \sum_{i=1}^{nb} \sum_{k=1}^{N_i} y_{ik} - \sum_{n=1}^{nn} s_n z_n \leq 0 \quad (23)$$

$$-C^T e_a + B r_a + f_a Z = 0 \quad (24)$$

$$t_{min_i} \sum_{k=1}^{N_i} A_{ik} y_{ik} \leq e_{a_i} \leq t_{max_i} \sum_{k=1}^{N_i} A_{ik} y_{ik} \quad (25)$$

$$y_{ik} \in \{0, 1\} \quad (26)$$

$$\sum_{k=1}^{N_i} y_{ik} \leq 1, \quad (27)$$

where $l = 1, \ldots, nc$, $j = 1, \ldots, 2nn$, $j_m = 1, \ldots, na$, $k = 1, \ldots, N_i$, $n = 1, \ldots, nn$ and $i = 1, \ldots, nb$.

The meanings of the parameters in this program are presented below:

nb	number of bars;
na	number of simple supports;
nn	number of nodes;
nc	number of loading conditions;
N_i	number of discrete sizes available for cross-sectional area of bar i;
A_{ik}	k-th discrete size for bar i;
C	$nb \times 2nn$ matrix of direction cosines relating bar forces with nodal directions;
B	$2nn \times na$ matrix of direction cosines relating nodal directions with nodal supports directions;
M	matrix $\left[C^T diag\left(\frac{E_i}{h_i}\right)\right]$;
E_i	Young's modulus of bar i;
h_i	length of bar i;
f_j^l	applied nodal loads in direction j for loading condition l;
$I(n)$	set of bars indices which occur in node n;
λ	safety factor;
$\|I(n)\|$	cardinal of set $I(n)$;
s_n	number of simple supports associated with node n;
Z	$2nn \times 2nn$ diagonal matrix, with z_{jj} equal to z_n of the node n associated to the direction j;
f_a	perturbed nodal load applied in all directions;
t_{min_i}, t_{max_i}	minimum and maximum stress in compression and tension, respectively, of bar i;
d_{min_i}, d_{max_i}	minimum and maximum elongation of bar i;
u_{min_j}, u_{max_j}	minimum and maximum nodal displacement in direction j.

The variables have the following meanings:

y_{ik_i}	$0-1$ variable stating whether the k-th discrete size for bar i is or not the cross-sectional area of bar i;
$e_{p_i}^l$	bar force of bar i for loading condition l;
$r_{p_m}^l$	plastic reaction in supports m for loading condition l;
$r_{e_m}^l$	elastic reaction in supports m for loading condition l;
d_i^l	deformation of bar i for loading condition l;
u_j^l	nodal displacement in the direction j for loading condition l;
$q_{ik_i}^l$	elongation of bar i corresponding to each discrete size k for bar i in loading condition l;
z_n	$0-1$ variable stating whether the node n exists or not;
e_{a_i}	bar force of bar i for the perturbed nodal load;
r_{a_m}	plastic reaction in supports m for the perturbed nodal load.

Thus the mixed–integer linear program (OPT) has

$$nc\times\left(4nn+5nb+2\sum_{i=1}^{nb}N_i\right)+3nb+4nn+1$$

constraints and

$$nc\times\left(2nb+2nn+2na+\sum_{i=1}^{nb}N_i\right)+\sum_{i=1}^{nb}N_i+nn+nb+na$$

variables.

3 Reduction to a Mathematical Program with Complementarity Constraints

In the previous section, the topological optimization model has been formulated as a mixed-integer linear program, which can be stated as

$$\begin{array}{ll} \text{PLI: } \textit{Minimize} & c^T x + d^T u \\ \quad \textit{subject to} & Ax + Bu = g \\ & Fu = h \\ & u \geq 0, \quad x_i \in \{0,1\}, \quad i = 1,\dots,n. \end{array} \tag{28}$$

As discussed in [2], this mixed integer program can be shown to be equivalent to the following Bilevel Program

$$\begin{array}{lll} \text{BL: } \textit{Minimize} & c^T x + d^T u & \\ \quad \textit{subject to} & Ax + Bu = g & \\ & 0 \leq x \leq e & \\ & u \geq 0 & \\ & Fu = h & \\ & e^T v = 0, \ v \geq 0 & \\ & \textit{Minimize} & -e^T v \\ & \textit{subject to} & v \leq x \\ & & v \leq e - x, \end{array} \tag{29}$$

where $e \in \mathbb{R}^n$ is a vector of ones.

By exploiting the Karush-Kuhn-Tucker conditions of the follower's problem (29), it is possible to reduce the BL problem into the following Mathematical Programming Problem with Equilibrium (or Complementarity) Constraints

$$
\text{MPEC:} \quad \begin{array}{ll} \textit{Minimize} & c^T x + d^T u \\ \textit{subject to} & \left.\begin{array}{l} Ax + Bu = g \\ 0 \le x \le e \\ u \ge 0 \\ Fu = h \\ \alpha + \beta = 1 \\ v + \tau - x = 0 \\ v + s + x = e \\ e^T v = 0 \\ \alpha,\ \beta,\ v,\ \tau,\ s \ge 0 \\ \alpha^T \tau = \beta^T s = 0 \end{array}\right\} \text{GLCP.} \end{array}
$$

The constraints of this MPEC constitute a Generalized Complementarity Problem (GLCP), which can be written in the form

$$
\begin{array}{l} Ew = q + Mz + Ny \\ w \ge 0,\ z \ge 0 \\ y \in K_y \\ z^T w = 0, \end{array} \tag{30}
$$

where

$$
K_y = \{y :\ y \ge 0,\ Cy = b\}\,. \tag{31}
$$

GLCP can be processed by a direct or an iterative method provided the matrices E and M satisfy some nice properties. In particular [17], if E is the identity matrix and M is a Positive Semi-Definite (PSD) matrix, then the GLCP can be solved by finding a stationary point of the following quadratic program

$$
\text{QP:} \begin{array}{ll} \textit{Minimize} & z^T w \\ \textit{subject to} & Ew = q + Mz + Ny \\ & z \ge 0,\ w \ge 0 \\ & y \in K_y. \end{array} \tag{32}
$$

Unfortunately, the GLCP under consideration does not satisfy this property, as the matrices E and M are not even square. An enumerative method [15, 21] is then required to process the GLCP. The algorithm searches for a solution of the GLCP by exploiting a binary tree that is constructed based on the dichotomy presented in the complementarity conditions $z_i w_i = 0$. At each node k, the algorithm computes a stationary point of a quadratic program QP(k) that is obtained from QP by adding the constraints

$$
\begin{array}{l} z_i = 0,\ i \in L_k \\ w_j = 0,\ j \in W_k, \end{array}
$$

where L_k and W_k are the set of the fixed variables at this node. The incorporation of such QP solver enables the enumerative algorithm to find a solution of the GLCP in a reasonable effort, even for large problems. In fact, a solution of the GLCP is exactly a stationary point of QP(k) such that $z^T w = 0$.

The enumerative algorithm faces difficulties when the GLCP is feasible (the linear constraints are consistent) but has no solution. In this case the last property does not hold and the method requires an exhaustive search in the tree to terminate.

As a final remark, it is important to add that the enumerative method can be implemented by using an active-set code such as MINOS [18]. A comparison of such an active-set implementation of the enumerative method with a reduced-gradient based version [15], shows that the former is in general more efficient to find a solution to the GLCP [21].

4 A Complementarity Active-Set Algorithm

The Complementarity Active-Set Algorithm [16] uses an active-set strategy [19] to find a stationary point of MPEC, that is, a solution satisfying the necessary first-order KKT conditions of the nonlinear program (NLP), that is obtained from MPEC (1) by considering the complementarity conditions $z_i w_i = 0$, $i = 1, \ldots, n$ as constraints. Thus this NLP has the following form

$$\text{NLP:} \quad \textit{Minimize} \quad c^T z + d^T y$$
$$\left.\begin{array}{rl} \textit{subject to} & Ew = q + Mz + Ny \\ & Cy = b \\ & z \geq 0, \quad w \geq 0, \quad y \geq 0 \\ & z_i w_i = 0, \; i = 1, \ldots, n. \end{array}\right] \text{GLCP} \qquad (33)$$

where $q \in \mathbb{R}^p$, $c, w, z \in \mathbb{R}^n$, $d, y \in \mathbb{R}^m$, $E, M \in \mathbb{R}^{p \times n}$, $N \in \mathbb{R}^{p \times m}$, $C \in \mathbb{R}^{l \times m}$ and $b \in \mathbb{R}^l$.

The algorithm consists essentially of using an active-set technique on the set of solutions of the GLCP given by the constraints of the MPEC. Thus at each iteration k, the iterates (w, z, y) satisfy the constraints of (1), and the set of the active constraints is given by

$$\left.\begin{array}{lll} Ew - Mz - Ny & = q & \\ Cy & = b & \\ w_i & = 0, & i \in L_w \subseteq \{1, \ldots, n\} \\ z_i & = 0, & i \in L_z \subseteq \{1, \ldots, n\} \\ y_i & = 0, & i \in L_y \subseteq \{1, \ldots, m\} \end{array}\right\} \qquad (34)$$

where L_z , L_y, and L_w are the sets of the currently active constraints corresponding to the nonnegative constraints on the variables z, y, and w, respectively and $L_z \cup L_w = \{1, \ldots, n\}$.

The active constraints (34) constitute a linear system of the form

$$D_k x = g^k,$$

where $x = (w^T, z^T, y^T)^T$ and $D_k \in \mathbb{R}^{t\times(2n+m)}$, with $t = l+p+ \mid L_w \mid + \mid L_z \mid + \mid L_y \mid$ and $\mid H \mid$ is the cardinality of the set H, where that p and l are the number of rows of the matrices A and $[E \quad -M - N]$ respectively.

The first-order optimality conditions for the problem

$$Minimize\{f(x): \ D_k x = g^k\}$$

can be written in the form

$$\begin{aligned} \nabla f(x) &= D_k^T \mu \\ D_k x &= g^k. \end{aligned}$$

In order to facilitate a unique set of Lagrange multipliers μ, the following condition is assumed to hold throughout the proposed procedure:

Nondegeneracy Assumption: $t \le 2n + m$ and $\text{rank}(D_k) = t$.

This hypothesis is not restrictive under the usual full row rank of the matrices C and $[E, -M, -N]$. Consequently, the active-set is always linearly independent. Furthermore, let us partition the Lagrange multipliers vector μ into three subvectors denoted by

β $\rightarrow$ subvector associated the first set of equality constraints in (34)
ϑ $\rightarrow$ subvector associated the second set of equality constraints in (34)
λ_i^x $\rightarrow$ subvector associated with $x_i = 0$ in the last three sets of equality constraints in (34).

The main steps of the complementary active-set algorithm are described below.

COMPLEMENTARITY ACTIVE-SET ALGORITHM - CASET

Step 0
Set $k = 1$ and find a solution x^k of the GLCP associated with MPEC. Let $D_k x = g^k$ be the set of active constraints at x^k and let L_y, L_z, and L_w be the index sets associated with the nonnegative active constraints $y_i = 0$, $z_i = 0$, and $w_i = 0$, respectively.

Step 1 *Optimality Conditions*

If x^k is not a stationary (KKT) point (see [4]) for the Equality Problem

$$\begin{aligned} \text{EP: } & Minimize \ f(x) \\ & subject \ \ to \ D_k x = g^k, \end{aligned}$$

then go to Step 2. Otherwise, there exists a unique μ such that

$$D_k^T \mu = \nabla f(x^k),$$

and two cases can occur:

1. If
$$\begin{array}{l} \lambda_i^y \geq 0 \text{ for all } i \in L_y \\ \lambda_i^z \geq 0 \text{ for all } i \in L_z \cap L_w \\ \lambda_i^w \geq 0 \text{ for all } i \in L_z \cap L_w, \end{array}$$
stop: x^k is a stationary point for MPEC.
2. If there exists at least one i such that
$$\begin{array}{l} \lambda_i^y < 0 \text{ for } i \in L_y \\ \text{or } \lambda_i^z < 0 \text{ for } i \in L_z \cap L_w \\ \text{or } \lambda_i^w < 0 \text{ for } i \in L_z \cap L_w, \end{array}$$
remove an active constraint $y_i = 0$, or $z_i = 0$, or $w_i = 0$, associated with the most negative Lagrange multiplier. Let $D_{k_i}x = g_i^k$ be the row removed from $D_k x = g^k$, and rearrange the rows of $D_k x = g^k$ in the following way
$$D_k = \begin{bmatrix} \bar{D}_k \\ D_{k_i} \end{bmatrix}, \quad g^k = \begin{bmatrix} \bar{g}^k \\ g_i^k \end{bmatrix}.$$
Find a direction d such that $\nabla f(x^k)^T d < 0$, $\bar{D}_k d = 0$, and $D_{k_i} d > 0$. Replace D_k by $\bar{D}_k$ and go to Step 3.

Step 2 *Determination of Search Direction*

Find a descent direction for f in the set of active constraints, i.e, find d such that
$$\begin{array}{r} \nabla f(x^k)^T d < 0 \\ D_k d = 0. \end{array}$$

Step 3 *Determination of Stepsize*

1. Find the largest value α_{max} of α such that
$$x^k + \alpha d \geq 0,$$
from
$$\alpha_{max} = min\left\{\frac{x_i^k}{-d_i} : d_i < 0, \quad i \notin (L_z \cup L_w \cup L_y)\right\}.$$
2. Compute $0 < \alpha_k \leq \alpha_{max}$ such that
$$x^k + \alpha_k d$$
provides a sufficient decrease for f using any line search technique [4].

If $\alpha_k = +\infty$, stop; MPEC is unbounded.

Step 4 *Update of iterate*

Compute
$$x^{k+1} = x^k + \alpha_k d.$$

If $\alpha_k = \alpha_{max}$, add to the active set the constraints $x_i \geq 0$ for which α_{max} was attained such that the nondegeneracy assumption remains true. Return to Step 1 with $k := k + 1$.

As it is shown in [16], this algorithm possesses global convergence to a Stationary Point of the MPEC under a nondegenerate condition. The algorithm can also be extended to deal with degenerate cases and can be implemented by using an active-set code, such as MINOS. Computational experience reported in [16] on the solution of MPECs, taken from different sources, indicates that the algorithm CASET is quite efficient to find stationary points for MPECs of moderate and even large dimensions.

5 A Sequential Linear Complementarity Algorithm - SLCP

In this section, we briefly describe a Sequential Linear Complementarity (SLCP) algorithm [14] that finds a global minimum of the MPEC, by computing a set of stationary points with strictly reducing objective function values. To do this, in each iteration k of the algorithm the objective function is replaced by the cut

$$c^T z + d^T y \leq \lambda_k,$$

where λ_k is a constant to be defined later. So in each iteration a GLCP(λ_k) of the form below is solved first:

$$\begin{array}{l} Ew = q + Mz + Ny \\ w \geq 0, \; z \geq 0 \\ y \in K_y \\ c^T z + d^T y \leq \lambda_k \\ z^T w = 0. \end{array} \tag{35}$$

Let $(\bar{w}, \bar{z}, \bar{y})$ be such a solution. Then algorithm CASET with this initial point is applied to find a stationary point of the MPEC. To guarantee that the algorithm moves toward a global minimum of the MPEC, the sequence of step lengths $\{\lambda_k\}$ must be strictly decreasing. An obvious definition for λ_k is as below

$$\lambda_k = c^T z^{k-1} - d^T y^{k-1} - \gamma \mid c^T z^{k-1} + d^T y^{k-1} \mid$$

where γ is a small positive number and $(w^{k-1}, z^{k-1}, y^{k-1})$ is the stationary point found in the previous iteration.

Now consider the GLCP(λ_k) given by (35). Then there are two possible cases as stated below.

(i) GLCP(λ_k) has a solution that has been found by the enumerative method discussed before.
(ii) GLCP(λ_k) has no solution.

In the first case the algorithm uses this solution to find the stationary point of the MPEC associated to this iteration. In the last case, the stationary point $(\bar{z}^{k-1}, \bar{y}^{k-1})$ computed in iteration $(k-1)$ is an ε-global minimum for the MPEC, where

$$\varepsilon = \gamma \mid c^T \bar{z}^{k-1} + d^T \bar{y}^{k-1} \mid \tag{36}$$

and γ is a small positive tolerance.

The steps of the SLCP algorithm are presented below.

SEQUENTIAL LINEAR COMPLEMENTARY ALGORITHM - SLCP

Step 0 Set $k = 0$. Let $\gamma > 0$ a positive tolerance and $\lambda_0 = +\infty$.

Step 1 Solve GLCP(λ_k). If GLCP(λ_k) has no solution, go to Step 2. Otherwise, let (w^k, z^k, y^k) be a solution of GLCP(λ_k). Apply CASET algorithm with this starting point to find a stationary point of MPEC. Let $(\bar{w}^k, \bar{z}^k, \bar{y}^k)$ be such a point. Let

$$\lambda_{k+1} = c^T \bar{z}^k + d^T \bar{y}^k - \gamma \mid c^T \bar{z}^k + d^T \bar{y}^k \mid .$$

Set $k = k + 1$ and repeat the step.

Step 2 If $k = 0$, MPEC has no solution. Otherwise, $(\bar{z}^{k-1}, \bar{y}^{k-1})$ is an ε-global optimal solution for the MPEC, where ε is given by (36) (it is usually a global minimum of the MPEC).

As discussed in Section 3, the enumerative method faces great difficulties to show that the last GLCP has no solution. So the SLCP algorithm is able to find a global minimum, but it has difficulties to establish that such a solution has been found. Computational experience presented in [13, 14, 15] confirms this type of behavior in practice. It is also important to add that the SLCP algorithm can be implemented by using an active-set code such as MINOS. In fact the SLCP algorithm only uses the enumerative and the CASET methods, which are both implemented by using this type of methodology.

6 Computational experience

In this section some computational experience is reported on the solution of some structural models introduced in [8] by exploiting the MPEC formulation and using the algorithms CASET and SLCP. All the computations have been performed on a Pentium IV 2.4GHz machine having 256 MB of RAM.

(I) Test Problems

In each test problem the corresponding initial structure consists of nodal points and bars and takes a similar form to the type mesh displayed in Figure 1.

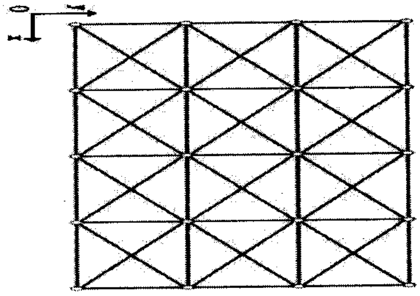

Fig. 1. Initial mesh

The main goal of this model is to find the set of included bars in the so–called optimal shape of the structure, which is given by the values of the $0-1$ variables x_i in the optimal solution of the problem.

Different types of sizes of initial meshes, as well as of applied nodal forces have been taken in consideration in the constitution of the test problems. Four sizes of initial meshes, M_i, $i = 0, \ldots, 3$, have been considered whose topologies are presented in Table 1 and that lead to five test problems PT0 to PT4, according to the following definitions:

- PT0 - mesh M0 and only one nodal load is applied ($f^1_{x_4} = 65$, $f^1_{y_4} = 0$).
- PT1, PT2 - mesh M1 and two types of applied nodal loads are applied. In PT1 is applied only one nodal load ($f^1_{x_8} = 0$, $f^1_{y_8} = -65$), while two nodal loads ($f^1_{x_8} = 0$, $f^1_{y_8} = -65$, $f^2_{x_9} - 40$, $f^2_{y_9} = -40$) are applied in PT2.
- PT3 - mesh M2 and two nodal loads are simultaneously applied ($f^1_{x_3}45.9619$, $f^1_{y_3} = -45.9619$, $f^1_{x_{12}} = 45.9619$, $f^1_{y_{12}} = 45.9619$).
- PT4 - mesh M3 and only one nodal load is applied ($f^1_{x_{23}} = 0$, $f^1_{y_{23}} = -65$).

In these definitions the following parameters are used:

$f^l_{x_n}$ nodal load in (*kN*) applied in node n in direction Ox for loads combination l;

$f^l_{y_n}$ nodal load in (*kN*) applied in node n in direction Oy for loads combination l.

In Table 1 are included the following notations:

nal dimension of the mesh in terms of number of nodal in Ox and Oy axes, respectively (in Figure 1, $nal = 5 \times 4$)

h_x total length (in m) to the Ox axis

h_y total length (in m) to the Oy axis

nb number of bars

nn number of nodes

na number of simple supports

S_i set of discrete sizes available for cross-sectional area of bar i (in cm^2)

N_i number of discrete sizes available for cross-sectional area of bar i

	MESH	h_x	h_y	nal	nb	nn	na	N_i	S_i
GROUP I	M0	4	3	2×2	6	4	3	1	3
	M1	8	6	3×3	20	9	3	1	3
	M2	6	9	3×4	29	12	8	1	3
	M3	16	12	5×5	72	25	3	1	3
GROUP II	SM1	8	6	3×3	20	9	3	2	0.5;3
	SM2	8	6	3×3	20	9	3	3	0.5;1;2
	SM3	6	9	3×4	29	12	8	2	0.5;3
	SM4	6	9	3×4	29	12	8	3	0.5;2;3

Table 1. Test Problems Meshes

In the first group of test problems, structures have been considered for which a unique discrete value is available for cross-sectional area of each bar. In the second group it is allowed that each bar of the structure assumes one of the values in a finite set S_i of discrete sizes available for its cross-sectional area. This last group leads to four additional test problems, assigned for ST1, ST2, ST3 and ST4, and whose associated initial meshes are SM1, SM2, SM3 and SM4, respectively. The meshes SM1 and SM2 have the same dimensions of the M1 mesh, while SM3 and SM4 have the same dimensions of the ones in M2. The nodal loads applied in ST1 and ST2 are the same as in PT1, while in ST3 and ST4 are the same as in PT3. The number of constraints (nr) and the number of variables (nv) of formulation OPT associated to these test problems are presented in Table 2.

		OPT	
	PROB	nr	nv
GROUP I	PT0	93	51
	PT1	273	136
	PT2	449	220
	PT3	387	205
	PT4	921	444
GROUP II	ST1	313	176
	ST2	353	216
	ST3	445	263
	ST4	503	321

Table 2. Dimensions of test problems

In all test problems the displacements and bars stress limits considered are $u_{max} = -u_{min} = 50cm$, $t_{max} = -t_{min} = 355MPa$, respectively and the partial safety factor λ is equal to 1.5.

(II) Solution of MPECs

This section reports the computational experience performed with the algorithms CASET and SLCP for the solution of MPECs associated with the integer linear program (OPT).

The dimensions of the resultant MPEC problems are included in Table 3, where nr, nv and nvc denote the number of constraints, number of variables and pairs of complementary variables, respectively.

		MPEC		
	PROB	nr	nv	nvc
GROUP I	PT0	124	111	30
	PT1	361	310	87
	PT2	537	394	87
	PT3	511	451	123
	PT4	1213	1026	291
GROUP II	ST1	461	470	147
	ST2	561	630	207
	ST3	656	683	210
	ST4	801	915	297

Table 3. Dimensions of MPEC test problems

Table 4 includes the performance of the integer program code OSL for finding a global minimum to the test problems [8]. In this table, as well as in the sequel, ND and NI are, respectively, the number of searched nodes and the number of iterations (pivot steps) performed by the process, T is the total CPU time in seconds for solving the optimization problem and OBJ. is the objective function value obtained by the algorithm. Note that for problem PT4, OSL code has not been able to terminate after 25000000 pivots steps.

The first computational experience has been performed with the CASET algorithm for finding a stationary point for the MPECs associated with the structural optimization problems and analyzing if this solution is near to the global optimal solution. Table 5 includes the performance of this algorithm on the solution of these test problems.

The numerical results clearly indicate that CASET algorithm has been able to find a structure with a volume close to the global optimal one. This is particularly evident for the problems with exactly one possible cross area for each bar. Furthermore this solution has been found in a quite small amount of effort as compared to that of the OSL code.

Table 6 includes the computational results achieved by the Sequential Complementary Algorithm on the solution of the test problems. In this table, besides the previously used parameters, IT represents the number of GLCP(λ_k) solved by algorithm SLCP, while NIS, TS and NDS are the number

Prob	OSL Ni	OSL T	OSL Nd	Obj (dm^3)
Pt0	53	0.04	7	3.60
Pt1	3033	0.69	311	10.80
Pt2	5579	1.77	497	12.90
Pt3	891143	325.64	82075	11.92
Pt4	>25000000	15018.78	347541	27.30
St1	64943	22.80	8132	7.05
St2	57473	30.01	10052	4.90
St3	4788682	3996.54	411084	6.29
St4	20606789	61486.08	1496081	5.46

Table 4. Computation of Global Minimum of Integer Program OPT by using the OSL code

Prob	CASET Ni	CASET T	CASET Obj
Pt0	116	0.03	3.60
Pt1	522	0.24	11.10
Pt2	961	0.44	14.10
Pt3	922	0.44	12.66
Pt4	9859	9.00	27.90
St1	726	0.36	7.10
St2	820	0.56	8.75
St3	1819	1.03	9.40
St4	1731	1.44	9.80

Table 5. Computation of a stationary point of MPEC by using the CASET algorithm

Prob	SLCP It	SLCP Ni	SLCP T	SLCP Nd	SLCP Nis	SLCP Ts	SLCP Nds	Obj.
Pt0	2	249	0.05	46	128	0.04	9	3.60
Pt1	2	5572	2.30	907	494	0.17	27	10.80
Pt2	3	28281	14.73	3828	1015	0.44	44	12.90
Pt3	6	63210	30.78	8278	23843	12.16	3417	11.92
Pt4(*)	4	140015	124.00	3884	39968	35.25	1073	27.30
St1	2	11370	5.09	1177	691	0.28	27	7.05
St2	16	35871	23.90	4723	12698	9.28	1779	4.90
St3	15	498688	279.89	34578	160949	86.52	9719	6.29
St4	16	1602271	1171.91	99962	421851	318.40	31144	5.46

Table 6. Application of SLCP algorithm to the structural problems

of pivot steps, the CPU time in seconds and the number of nodes searched until the optimal solution is obtained, respectively. Moreover, the notation (*)

is used in problem PT4 to indicate that the solution of the last GLCP was interrupted because the maximum limit of 100000 pivot steps was exceeded.

A comparison between the SLCP algorithm and the code OSL shows that the latter procedure performs better for problems of smaller dimensions. However, as the dimension increases the SLCP algorithm becomes more efficient to obtain a global minimum. It is important to add that the SLCP algorithm computes stationary points of the MPEC with strictly decreasing objective function value. Since each one of these stationary points corresponds to a feasible solution of the zero-one integer programming formulation of the structural model, then the engineer is able to receive a number of structures (equal to the number of iterations of the algorithm SLCP) in a reasonable amount of time. For instance for problem ST4 the algorithm SLCP requires only 421851 pivot steps to give the engineer 16 structures including the one given by the CASET method and the global optimal structure.

7 Conclusions

In this paper we have investigated the solution of a zero-one integer program associated with a structural model by using two MPEC techniques. A Complementarity Active-Set (CASET) algorithm for finding a stationary point of a MPEC and a Sequential Linear Complementarity (SLCP) algorithm for computing a global minimum have been considered in this study. Numerical results of some experiments with these techniques show that both procedures are in general efficient for their purposes. We believe that the results shown in this paper may influence the use of MPEC algorithms to process integer programming problems. This is a subject of future research.

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