A Block Active Set Algorithm with Spectral Choice Line Search for the Symmetric Eigenvalue Complementarity Problem

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Abstract

In this paper, we address the solution of the symmetric eigenvalue complementarity problem (EiCP) by treating an equivalent reformulation of finding a stationary point of a fractional quadratic program on the unit simplex. The spectral projected-gradient (SPG) method has been recommended to this optimization problem when the dimension of the symmetric EiCP is large and the accuracy of the solution is not a very important issue. We suggest a new algorithm which combines elements from the SPG method and the block active set method, where the latter was originally designed for box constrained quadratic programs. In the new algorithm the projection onto the unit simplex in the SPG method is replaced by the much cheaper projection onto a box. This can be of particular advantage for large and sparse symmetric EiCPs. Global convergence to a solution of the symmetric EiCP is established. Computational experience with medium and large symmetric EiCPs is reported to illustrate the efficacy and efficiency of the new algorithm.

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1. Introduction

The eigenvalue complementarity problem (EiCP) consists in finding $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\}$ satisfying

$$(\lambda B - A)x \geq 0, \quad x \geq 0, \quad x^\top(\lambda B - A)x = 0, \tag{1}$$

where $A, B \in \mathbb{R}^{n\times n}$ are given matrices with B being positive definite. Equivalently, we may seek $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n$ so that

$$(\lambda B - A)x \geq 0, \quad x \geq 0, \quad x^\top(\lambda B - A)x = 0, \quad e^\top x = 1 \tag{2}$$

holds with $e := (1, \ldots, 1)^\top \in \mathbb{R}^n$. If a pair (λ, x) solves EiCP, then λ is called complementary eigenvalue and x complementary eigenvector associated to λ. The EiCP has been introduced in [31] as a generalization of the classical eigenvalue problem to a closed and convex cone. To our knowledge, a first interesting engineering application of the EiCP was described in [28]. In recent years, many other papers appeared in the literature describing theoretical results, algorithms, and applications for EiCPs as well as some extensions [1, 2, 9, 10, 17, 18, 20, 21, 22, 25, 27, 29, 30, 32, 33, 38].

In this paper, we are dealing with symmetric EiCPs [30], i.e., the matrices A and B are both symmetric. For any $x \in \mathbb{R}^n \setminus \{0\}$, let

$$f(x) := -\frac{x^\top Ax}{x^\top Bx}$$

denote the negative generalized Rayleigh quotient associated to EiCP (1). It was shown in [30, 34] that any stationary point x^* of the minimization problem

$$f(x) \to \min \quad \text{s.t.} \quad x \in \Delta := \{x \in \mathbb{R}^n \mid e^\top x = 1, x \geq 0\} \tag{3}$$

with $\lambda^* := -f(x^*)$ provides a solution of (2), and vice versa. This problem can also be regarded as a fractional quadratic program over the unit simplex

Δ. Nonlinear programming algorithms [26] may be used to solve the symmetric EiCP by computing a stationary point of (3). In particular, the use of the spectral projected-gradient (SPG) method [5, 6] was suggested in [20]. In each step of the SPG method the direction $-\eta\nabla f(x)$ equipped with the spectral choice line search parameter η [4] is projected onto the simplex Δ. In this way, a feasible descent direction is obtained and f is minimized along this direction. This projection onto Δ has the worst-case complexity of $O(n^2)$ per step and can be done quite efficiently by one of the methods described in [19, 37]. The SPG method incorporating one of these projection techniques was reported [20] to perform well for large-scale EiCPs, particularly when the accuracy of the computed solution is not a very important issue.

In this paper, we introduce a new algorithm which combines ideas from the SPG method and from the block active set (BAS) method described in [16]. The new spectral block active set algorithm (Spectral BAS) employs in each iteration a block active set strategy for forecasting the active set at some stationary point. Several components of the search direction are determined in this way. The remaining components of the search direction are computed by the SPG strategy mentioned before. However, instead of projecting onto Δ, only a projection onto the nonnegative orthant is needed. An exact line search and a normalization step complete the algorithm and guarantee that the iterates stay within the simplex Δ. Hence, the Spectral BAS algorithm only requires cheap projections of complexity $O(n)$ onto the nonnegative orthant, which can be of particular interest for large-scale sparse EiCPs. Computational experience reported in this paper shows that the algorithm seems to be quite efficient for the solution of symmetric EiCPs associated to the maximum clique problem [7]. This may have important implications on the design of new algorithms for this difficult problem which are based on the solution of EiCPs. The algorithm is also efficient to solve symmetric EiCPs with unstructured sparse matrices if the accuracy is not too at stake, but may face some difficulties for getting more accurate solutions. A new preprocessing technique is also introduced that improves the efficiency of the Spectral BAS algorithm in practice. Furthermore, the Spectral BAS algorithm is competitive with the SPG method and seems to be more efficient for large-scale EiCPs. This is mainly due to the cheaper projection technique used by the Spectral BAS algorithm.

The paper is organized as follows. We briefly review a characterization of stationary points of the minimization problem (3) in Section 2. Then, Section 3 provides a detailed description of the new Spectral BAS algorithm including its well-definedness and important basic properties. Global convergence of this algorithm is shown in Section 4. Computational experience with the Spectral BAS algorithm and a comparison with the SPG method are reported in Section 5. Finally, some conclusions are included in the last section of the paper.

Notation. For a vector $x \in \mathbb{R}^n$ and an index set $J \subset I := \{1, \dots, n\}$ the vector x_J consists of all components x_j of x with $j \in J$. The Euclidean projection of some $z \in \mathbb{R}^n$ onto the simplex Δ is denoted by $P_\Delta(z)$. If $y \in \mathbb{R}^q$ is projected onto the nonnegative orthant $\mathbb{R}^q_+$ we simply write y_+ to denote the result of this projection.

2. Properties of the fractional program

The unit simplex Δ is a nonempty, closed, and convex set. Therefore, $x^* \in \Delta$ is a stationary point of (3) if and only if the first-order necessary optimality condition for (3) are satisfied at x^*, i.e., if

$$\nabla f(x^*)^\top (x - x^*) \geq 0 \quad \text{for all } x \in \Delta. \tag{4}$$

By the positive definiteness of B, the function $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is continuously differentiable with

$$\nabla f(x) = \frac{2}{x^\top Bx} \left(\frac{x^\top Ax}{x^\top Bx} Bx - Ax \right) \tag{5}$$

and it can be easily verified that

$$x^\top \nabla f(x) = 0 \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{6}$$

Thus, instead of (4), a stationary point x^* of (3) can be characterized by $x^* \in \Delta$ and

$$\nabla f(x^*)^\top x \geq 0 \quad \text{for all } x \in \Delta. \tag{7}$$

Because all vectors of the canonical basis of $\mathbb{R}^n$ belong to Δ, condition (7) implies $\nabla f(x^*) \geq 0$. This, $x^* \geq 0$, and (6) yield that x^* is stationary for (3) if and only if x^* belongs to Δ and satisfies

$$\min\{x, \nabla f(x)\} = 0. \tag{8}$$

With the index sets

$$L^* := \{i \in I \mid x_i^* = 0\} \quad \text{and} \quad F^* := \{i \in I \mid x_i^* > 0\},$$

we therefore see that x^* is stationary for (3) if and only if $x^* \in \Delta$,

$$\nabla_i f(x^*) \geq 0 \quad \text{for all } i \in L^*, \quad \text{and} \quad \nabla_i f(x^*) = 0 \quad \text{for all } i \in F^*. \tag{9}$$

We conclude this section with an obvious but useful property of the function f.

Proposition 1. *For any $x \in \mathbb{R}^n \setminus \{0\}$ and any $\mu \neq 0$,*

$$f(\mu x) = f(x)$$

holds.

3. Spectral Block Active Set algorithm

Based on the BAS method [16] for dealing with box constrained quadratic programs and on the SPG method in [6, 20] we first present the ingredients of the new Spectral BAS algorithm for computing a stationary point of (3). In addition to combining ideas of both methods, we suggest to normalize the iterates in each step to stay in the unit simplex Δ. Like the methods in [6, 16, 20], the Spectral BAS algorithm uses a line search which will be completely described within this section. Moreover, we also provide some important basic properties of the new algorithm.

For a given point $x^k \in \Delta$, the BAS method [16] uses a strategy for forecasting the indices of active variables at some stationary point. Similarly, according to (9), the index set L^* defined above is estimated by the index set

$$L(x^k) := \{i \in I \mid x_i^k \leq \beta \nabla_i f(x^k)\}, \tag{10}$$

where β denotes a given positive number. For x^k sufficiently close to x^* we can easily see that

$$\{i \in I \mid \nabla_i f(x^*) > 0\} \subset L(x^k) \subset L^*$$

must be valid. To exploit the forecast $L(x^k)$ the corresponding components d_L^k of the search direction d^k are set as

$$d_L^k := -x_L^k. \tag{11}$$

With

$$L := L(x^k) \text{ and } F := F(x^k) := I \setminus L(x^k),$$

we have

$$d^k = \begin{pmatrix} d_L^k \\ d_F^k \end{pmatrix}. \tag{12}$$

The remaining subvector d_F^k will guarantee that the new iterate $x^k + \delta d^k$ stays nonnegative for all $\delta \in [0, 1]$ and that the complete search direction d^k is a descent direction of the objective function f at x^k. To this end, the BAS method in [16] allows to exploit some second order information by solving a box constrained positive definite quadratic program. Doing similar things for problem (8) may (regardless of a theoretical foundation) be computationally expensive for large sparse problems. Therefore, we suggest to apply the technique of the SPG method and to define d_F^k by

$$d_F^k := \left(x_F^k - \eta_k \nabla_F f(x^k)\right)_+ - x_F^k, \tag{13}$$

where η_k is a positive parameter. The definition of η_k is similar to the SPG method for symmetric EiCPs in [20] and will be detailed below. Recall that the search direction in the latter paper is

$$d_{SPG}^k := P_\Delta(x^k - \eta_k \nabla f(x^k)) - x^k.$$

Its computation has the worst-case complexity of $O(n^2)$. In contrast to this, computing d^k according to (11) and (13) has complexity $O(n)$. With

$$y^k := x_F^k - \eta_k \nabla_F f(x^k),$$

a well-known property of the projection onto a closed convex set provides

$$\left(y^k - y^k_+\right)^\top \left(z - y^k_+\right) \leq 0 \qquad \text{for all } z \in \mathbb{R}^{|F|}_+. \tag{14}$$

Moreover, the definition of d^k_F in (13) yields

$$d^k_i = \begin{cases} -x^k_i, & \text{if } x^k_i - \eta_k \nabla_i f(x^k) < 0, \\ -\eta_k \nabla_i f(x^k), & \text{if } x^k_i - \eta_k \nabla_i f(x^k) \geq 0, \end{cases} \qquad \text{for all } i \in F(x^k). \tag{15}$$

Before stating the Spectral BAS algorithm, the choice of the parameter η_k is described. Its definition stems from the Barzilai and Borwein step length [4] for the steepest-descent method in unconstrained optimization. The definition of η_k below follows the way for the SPG method in [5].

Let $\eta_{\min} \in (0, 1)$ be given, set

$$\eta_{\max} := \eta^{-1}_{\min}, \tag{16}$$

and choose

$$\eta_0 \in [1, \eta_{\max}]. \tag{17}$$

Then, for $k = 1, 2, 3, \ldots$ and with

$$s^k := x^k - x^{k-1} \qquad \text{and} \qquad w^k := \nabla f(x^k) - \nabla f(x^{k-1}),$$

the spectral parameter η_k is defined by

$$\eta_k := \begin{cases} \eta_{\max}, & \text{if } (s^k)^\top w^k \leq 0, \\ \min\left\{\eta_{\max}, \max\left\{\eta_{\min}, \dfrac{(s^k)^\top s^k}{(s^k)^\top w^k}\right\}\right\}, & \text{otherwise.} \end{cases} \tag{18}$$

This definition of the parameter η_k ensures that

$$0 < \eta_{\min} \leq \eta_k \leq \eta_{\max} \tag{19}$$

holds for any $k \in \{0, 1, 2, \ldots\}$. Now, we are in the position to establish the new method for computing a stationary point of problem (3).

Algorithm 1. (Spectral Block Active Set Algorithm)

S1: Choose $x^0 \in \Delta$, $\beta > 0$, $\eta_{\min} \in (0, 1)$, $\varepsilon \geq 0$ and set $k := 0$.

S2: Compute η_k by (16)–(18) and d^k according to (11)–(13).

S3: If $||d^k||_2 \leq \varepsilon$, then STOP.

S4: Compute a step size $\delta_k \in [0,1]$ (exact line search) so that

$$f(x^k + \delta_k d^k) \leq f(x^k + \delta d^k) \quad \text{for all } \delta \in [0,1]. \tag{20}$$

S5: Set $\hat{x}^{k+1} := x^k + \delta_k d^k$ and compute $\mu_k := e^\top \hat{x}^{k+1}$.

S6: Set $x^{k+1} := \dfrac{1}{\mu_k}\hat{x}^{k+1},\ k := k+1$, and goto S2.

In the remainder of this section we first show basic properties of Algorithm 1 that are important for proving its global convergence in Section 4. Thereafter, the computation of the Cauchy step size in step S4 of Algorithm 1 is dealt with.

Lemma 1. *Suppose that $x^k \in \Delta$ is determined by Algorithm 1 for some $k \in \{0, 1, 2, \ldots\}$. Then, $\nabla f(x^k)$ and d^k computed according to (11)–(13) are well defined. Moreover, the following assertions are valid:*

(i) $\nabla f(x^k)^\top d^k \leq -\gamma \|d^k\|_2^2$ *with* $\gamma := \min\{\beta^{-1}, \eta_{\min}\}$,

(ii) $x^k + \delta d^k \geq 0$ *for all* $\delta \in [0,1]$,

(iii) $x^k + \delta d^k \neq 0$ *for all* $\delta \in \mathbb{R}$,

(iv) if $d^k = 0$, *then* x^k *is a stationary point of* (3), *and vice versa,*

(v) if $\|d^k\|_2 > \varepsilon$, *then, for steps S4–S6, it holds that*

- *the step size* δ_k *is well defined,*
- $\mu_k > 0$, $x^{k+1} \in \Delta$, *and* $f(x^{k+1}) = f(\hat{x}^{k+1})$.

Proof: By $x^k \in \Delta$ it follows that $0 \leq x^k \neq 0$. Thus, $f(x^k)$ and $\nabla f(x^k)$ are well defined since B is positive definite. Moreover, d^k can be easily determined by (11)–(13).

(i) Clearly,

$$d_i^k = 0 \quad \text{implies} \quad \nabla_i f(x^k) d_i^k \leq -\gamma (d_i^k)^2 \qquad \text{for all } i \in I. \tag{21}$$

Suppose now that $d_i^k \neq 0$ for some $i \in I$. Then, (10), (11), and $x_i^k \geq 0$ yield

$$\beta \nabla_i f(x^k) \geq x_i^k = -d_i^k > 0 \qquad \text{for all } i \in L(x^k).$$

Multiplying this by $d_i^k < 0$ provides

$$\nabla_L f(x^k)^\top d_L^k \leq -\frac{1}{\beta}\|d_L^k\|_2^2. \tag{22}$$

The definition (13) of d_F^k implies

$$d_F^k = y_+^k - x_F^k.$$

Taking into account the latter, $y^k = x_F^k - \eta_k \nabla_F f(x^k)$, and choosing $z := x_F^k \geq 0$ within (14), we obtain

$$(d_F^k + \eta_k \nabla_F f(x^k))^\top d_F^k \leq 0.$$

Using (17), (18), and (16), we get

$$\nabla_F f(x^k)^\top d_F^k \leq -\frac{1}{\eta_k}\|d_F^k\|_2^2 \leq -\frac{1}{\eta_{max}}\|d_F^k\|_2^2 = -\eta_{\min}\|d_F^k\|_2^2. \tag{23}$$

Thus, in view of (21) and (22), assertion (i) follows with $\gamma := \min\{\beta^{-1}, \eta_{\min}\}$.

(ii) The assertion is obviously implied by $x^k \geq 0$ and $x^k + d^k \geq 0$. The latter holds by the definition of d^k according to (11)–(13).

(iii) Let us suppose the contrary, i.e., there is $\hat{\delta} \in \mathbb{R}$ so that $x^k + \hat{\delta} d^k = 0$. Since $x^k \in \Delta$ it follows that $x^k \neq 0$ and, in turn, that $\hat{\delta} \neq 0$ and

$$d^k = \frac{-x^k}{\hat{\delta}} \neq 0.$$

Therefore, assertion (i) implies that

$$f(x^k + \bar{\delta} d^k) = f\left(x^k\left(1 - \frac{\bar{\delta}}{\hat{\delta}}\right)\right) < f(x^k)$$

is valid for some $0 < \bar{\delta} < |\hat{\delta}|$. Because $\left(1 - \frac{\bar{\delta}}{\hat{\delta}}\right) \notin \{0, 1\}$, the latter contradicts Proposition 1. Thus, assertion (iii) is valid.

(iv) First, we suppose that $d^k = 0$. Then, (10) and (11) imply

$$0 = -d_i^k/\beta = x_i^k/\beta \leq \nabla_i f(x^k) \qquad \text{for all } i \in L(x^k). \tag{24}$$

For the remaining indices, (15) and $\eta_k \geq \eta_{\min} > 0$ by (19) provide

$$0 = -d_i^k/\eta_k = x_i^k/\eta_k < \nabla_i f(x^k), \qquad \text{if } i \in F(x^k) \text{ with } x_i^k < \eta_k \nabla_i f(x^k),$$

$$0 = -d_i^k/\eta_k = \nabla_i f(x^k), \qquad \text{if } i \in F(x^k) \text{ with } x_i^k \geq \eta_k \nabla_i f(x^k).$$

This and (24) yield $\min\{x^k, \nabla f(x^k)\} = 0$. Thus, according to (8), $x^k \in \Delta$ is a stationary point of (3).

Conversely, let x^k be a stationary point of (3). Then, (8) implies $\nabla f(x^k) \geq 0$. Paying attention to assertion (ii) for $\delta = 1$, (6), and assertion (i), we obtain

$$0 \leq \nabla f(x^k)^\top (x^k + d^k) = \nabla f(x^k)^\top d^k \leq -\gamma \|d^k\|_2^2 \leq 0. \tag{25}$$

This shows $d^k = 0$.

(v) If $\|d^k\|_2 > \varepsilon$, then Algorithm 1 does not stop at step S3 and goes to step S4. Since $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is continuously differentiable (see Section 2) we obtain from assertion (iii) that the function $\varphi : \mathbb{R} \to \mathbb{R}$ with

$$\varphi(\delta) := f(x^k + \delta d^k), \tag{26}$$

is well defined and continuously differentiable as well. Hence, φ has (at least) one minimizer δ_k (step size) within the compact set $[0, 1]$.

By assertions (ii) and (iii), we see that $0 \neq \hat{x}^{k+1} = x^k + \delta_k d^k \geq 0$ and $\mu_k = e^\top \hat{x}^{k+1} > 0$. Thus,

$$e^\top x^{k+1} = \frac{1}{\mu_k} e^\top \hat{x}^{k+1} = 1$$

is valid which, due to assertion (ii), implies $x^{k+1} \in \Delta$. Finally, Proposition 1 ensures that $f(x^{k+1}) = f(\frac{1}{\mu_k}\hat{x}^{k+1}) = f(\hat{x}^{k+1})$. $\square$

Note that Lemma 1 particularly states that d^k is a descent direction of the objective function f at x^k if and only if x^k is not a stationary point. Furthermore, the termination rule (step S3) is motivated by assertion (iv) of the lemma. The next result follows easily by induction and application of Lemma 1. Therefore, its proof is omitted.

Theorem 1. *Algorithm 1 is well defined and generates a sequence $\{x^k\} \subset \Delta$. If $\varepsilon = 0$ and if the sequence $\{x^k\}$ is finite (the algorithms stops in step S3), then the last element of $\{x^k\}$ solves EiCP.*

Global convergence of Algorithm 1 is the subject of Section 4.

At the end of the current section, the computation of the step length δ_k in step S4 of Algorithm 1 will be dealt with, where we follow the way introduced in [20]. According to step S4 of the algorithm, δ_k is a solution of the optimization problem

$$\varphi(\delta) \to \min \qquad \text{s.t. } 0 \leq \delta \leq 1, \tag{27}$$

where the function $\varphi : \mathbb{R} \to \mathbb{R}$ is given by (26). Assertion (v) in Lemma 1 tells us that this problem has at least one solution. Moreover, the proof of assertion (v) shows that φ is continuously differentiable. Hence, to find a solution of (27) we should consider zeros of φ'. To formulate a corresponding result, let the values a_0, a_1, and a_2 be defined by

$$a_0 := a_0(d^k, x^k), \quad a_1 := a_1(d^k, x^k), \quad a_2 := a_2(d^k, x^k)$$

with

$$\begin{aligned} a_0(d,x) &:= (d^\top Ax)(x^\top Bx) - (d^\top Bx)(x^\top Ax), \\ a_1(d,x) &:= (d^\top Ad)(x^\top Bx) - (d^\top Bd)(x^\top Ax), \\ a_2(d,x) &:= (d^\top Ad)(x^\top Bd) - (d^\top Bd)(x^\top Ad). \end{aligned}$$

Lemma 2. *Each solution of $\varphi'(\delta) = 0$ is a root of the quadratic equation*

$$a_0 + a_1\delta + a_2\delta^2 = 0, \tag{28}$$

and vice versa.

Proof: Defining

$$x := x^k, \quad d := d^k, \quad \text{and } b := \left((x+\delta d)^\top B(x+\delta d)\right)^2,$$

we see by the positive definiteness of B and assertion (iii) of Lemma 1 that b is positive. The continuous differentiability of φ and (5) lead to $\varphi'(\delta) =$

$d^\top \nabla f(x+\delta d)$. Now, simple calculations provide

$$\begin{aligned}
\varphi'(\delta) &= \frac{2}{b}\big((x+\delta d)^\top A(x+\delta d)d^\top B(x+\delta d) \\
&\quad -d^\top A(x+\delta d)(x+\delta d)^\top B(x+\delta d)\big) \\
&= \frac{2}{b}\Big((d^\top Bx)(x^\top Ax)-(d^\top Ax)(x^\top Bx) \\
&\quad +\delta\left((d^\top Bd)(x^\top Ax)-(d^\top Ad)(x^\top Bx)\right) \\
&\quad +\delta^2\left((d^\top Bd)(d^\top Ax)-(d^\top Ad)(d^\top Bx)\right)\Big) \\
&= -\frac{2}{b}(a_0+a_1\delta+a_2\delta^2).
\end{aligned}$$

Thus, $\varphi'(\delta)=0$ if and only if δ satisfies (28). □

Lemma 3. *Let x^k and d^k be generated by Algorithm 1. Moreover, if the quadratic equation* (28) *has two (possibly equal) solutions, they are denoted by δ^1 and δ^2 with $\delta^1 \le \delta^2$. Then, if $\|d^k\|_2 > \varepsilon$, the step size δ_k in step S4 can be determined by*

$$\delta_k := \begin{cases} 1, & \textit{if there is no solution of } (28) \textit{ in } [0,1], \\ \operatorname{argmin}\{\varphi(\delta^1),\varphi(1)\}, & \textit{otherwise.} \end{cases}$$

Proof: If $\|d^k\|_2 > 0$ we obtain from assertion (i) of Lemma 1 that

$$\varphi'(0) = \nabla f(x^k)^\top d^k < 0,$$

i.e., d^k is a descent direction of f at x^k. Thus, any minimizer δ_k of problem (27) belongs to the interval $(0,1]$ and satisfies $\varphi'(\delta_k)=0$ or $\delta_k=1$. If there is no solution of (28) within $[0,1]$ then, $\delta_k=1$ is the only minimizer of problem (27). Otherwise, if (28) has at least one solution in $[0,1]$ and if $\delta^1<\delta^2$, then $\varphi(\delta^1)<\varphi(\delta^2)$. Moreover, in the case when $\delta_1=\delta_2\in[0,1]$ we have $\varphi(\delta_1)\ge\varphi(1)$. This provides the formula for δ_k. □

4. Global convergence

In order to prove global convergence the following lemma is helpful.

Lemma 4. *Let $\{x^k\}$ and $\{d^k\}$ be infinite sequences generated by Algorithm 1, $J \subset \mathbb{N}$ an infinite subsequence of indices, and $\bar{x} \in \mathbb{R}^n$ so that*

$$\bar{x} = \lim_{k \in J} x^k \quad \textit{and} \quad \lim_{k \in J} d^k = 0. \tag{29}$$

Then, $\bar{x}$ is a stationary point of the minimization problem (3).

Proof: Because $I = \{1, ..., n\}$ is a finite index set the number of different subsets, in particular the number of different subsets $L(x^k) \subset I$ generated for infinitely many indices $k \in J$, is finite, too. Consequently, there are an infinite subset $J_1 \subset J \subset \mathbb{N}$ and index sets $\mathcal{L}, \mathcal{F} \subset I$ such that

$$L(x^k) = \mathcal{L} \quad \text{and} \quad F(x^k) = \mathcal{F} = I \setminus L(x^k) = I \setminus \mathcal{L} \qquad \text{for all } k \in J_1.$$

Since f is continuously differentiable on $\mathbb{R}^n \setminus \{0\}$, we obtain from (29), (11), and (10) that

$$0 = -\lim_{k \in J_1} d^k_{\mathcal{L}} = \lim_{k \in J_1} x^k_{\mathcal{L}} \leq \beta \lim_{k \in J_1} \nabla_{\mathcal{L}} f(x^k) = \nabla_{\mathcal{L}} f(\bar{x}), \quad \bar{x}_{\mathcal{L}} = 0. \tag{30}$$

For the remaining indices $i \in \mathcal{F}$, (13) provides

$$y^k_+ = (x^k_{\mathcal{F}} - \eta_k \nabla_{\mathcal{F}} f(x^k))_+ = d^k_{\mathcal{F}} + x^k_{\mathcal{F}} \qquad \text{for all } k \in J_1. \tag{31}$$

To show that $\bar{x}$ is a stationary point of problem (3) let $x \in \Delta$ be arbitrarily chosen. Then, clearly $x_{\mathcal{L}} \geq 0$, $x_{\mathcal{F}} \geq 0$. Using (14) with $z := x_{\mathcal{F}}$ and (31), we obtain for all $k \in J_1$

$$(y^k - y^k_+)^\top (x_{\mathcal{F}} - y^k_+) = -\left(\eta_k \nabla_{\mathcal{F}} f(x^k) + d^k_{\mathcal{F}}\right)^\top \left(x_{\mathcal{F}} - (d^k_{\mathcal{F}} + x^k_{\mathcal{F}})\right) \leq 0. \tag{32}$$

From (19), it follows that there are $\bar{\eta} \in [\eta_{\min}, \eta_{\max}]$ and an infinite subset J_2 of J_1 so that

$$\lim_{k \in J_2} \eta_k = \bar{\eta} > 0.$$

This, (29), and (32) yield

$$0 \leq \lim_{k \in J_2} (d^k_{\mathcal{F}} + \eta_k \nabla_{\mathcal{F}} f(x^k))^\top (x_{\mathcal{F}} - (x^k_{\mathcal{F}} + d^k_{\mathcal{F}})) = \bar{\eta} \nabla_{\mathcal{F}} f(\bar{x})^\top (x_{\mathcal{F}} - \bar{x}_{\mathcal{F}}). \tag{33}$$

Since, by Theorem 1, $x^k \in \Delta$ holds for all $k \in \mathbb{N}$, it follows from (29) that $\bar{x} \in \Delta$. Thus, (30) and (33) imply

$$\nabla f(\bar{x})^\top (x-\bar{x}) = \nabla_{\mathcal{L}} f(\bar{x})^\top (x_{\mathcal{L}} - \underbrace{\bar{x}_{\mathcal{L}}}_{=0}) + \underbrace{\nabla_{\mathcal{F}} f(\bar{x})^\top (x_{\mathcal{F}} - \bar{x}_{\mathcal{F}})}_{\geq 0} \geq \nabla_{\mathcal{L}} f(\bar{x})^\top x_{\mathcal{L}} \geq 0.$$

Since $x \in \Delta$ was arbitrarily chosen, $\bar{x} \in \Delta$ fulfills condition (4) and, hence, is a stationary point of problem (3). □

Now, it is possible to prove global convergence of the Spectral BAS Algorithm.

Theorem 2. *Let $\{x^k\}$ be an infinite sequence generated by Algorithm 1. Then, the sequence $\{x^k\}$ has at least one accumulation point. Moreover, if $\varepsilon = 0$, then each accumulation point of this sequence is a stationary point of problem* (3) *and, thus, solves EiCP.*

Proof: Due to Theorem 1, the infinite sequence $\{x^k\}$ generated by Algorithm 1 belongs to the compact set Δ. Therefore, $\{x^k\}$ has at least one accumulation point in Δ. Let $\bar{x}$ denote such an accumulation point. Then, an infinite set $J \subset \mathbb{N}$ exists so that

$$\lim_{k \in J} x^k = \bar{x}. \tag{34}$$

In order to apply Lemma 4, we show that $\{d^k\}$ converges to 0. To this end, we first verify that $\{d^k\}$ is bounded. From the definition (11) of the search direction d_L^k and by $\{x^k\} \subset \Delta$ it is clear that

$$\|d_L^k\|_2 \leq \|d_L^k\|_1 = \|x_L^k\|_1 \leq \|x^k\|_1 = 1 \qquad \text{for all } k \in \mathbb{N}. \tag{35}$$

For the remaining indices in $F(x^k) = I \setminus L(x^k)$, (15) and $\eta_k \in [\eta_{\min}, \eta_{\max}]$ by (19) yield

$$\|d_F^k\|_2 \leq \max\{\|x_F^k\|_2, \|\eta_k \nabla_F f(x^k)\|_2\} \leq M \qquad \text{for all } k \in \mathbb{N},$$

where $M := \eta_{\max} \max\{\|\nabla f(x)\|_2 \,|\, x \in \Delta\} + 1$ is well defined since f is continuously differentiable on some open set containing the compact set Δ. Therefore, using (35), we have

$$\|d^k\|_2 \le M+1 \qquad \text{for all } k \in \mathbb{N}. \tag{36}$$

For proving that $\lim_{k\to\infty} d^k = 0$ let us first choose a constant $\sigma \in (0,1)$. Then, the compact set

$$\mathcal{C} := \left\{ x \in \mathbb{R}^n \mid (1-\sigma)\frac{1}{\sqrt{n}} \le \|x\|_2 \le M+2 \right\}$$

does not contain the origin so that f is continuously differentiable on some open set containing $\mathcal{C}$. We now show that

$$x^k + \delta d^k \in \mathcal{C} \qquad \text{for all } k \in \mathbb{N} \text{ and all } \delta \in [0,\sigma]. \tag{37}$$

Since assertion (ii) of Lemma 1 yields $-x^k \le d^k$, we obtain

$$0 \le (1-\sigma)x^k = x^k - \sigma x^k \le x^k + \delta d^k \qquad \text{for all } \delta \in [0,\sigma].$$

Taking into account $x^k \in \Delta$ and (36),

$$(1-\sigma)\frac{1}{\sqrt{n}} \le (1-\sigma)\|x^k\|_2 \le \|x^k + \delta d^k\|_2$$

and

$$\|x^k + \delta d^k\|_2 \le \|x^k\|_2 + \|d^k\|_2 \le \|x^k\|_1 + \|d^k\|_2 \le M+2$$

follow for all $k \in \mathbb{N}$ and all $\delta \in [0,\sigma]$. Hence, (37) is valid. Because $f : \mathbb{R}^n \setminus \{0\}$ is also twice continuously differentiable the compactness of $\mathcal{C}$ and $0 \notin \mathcal{C}$ ensure that

$$\theta := \max\left\{ \|\nabla^2 f(x)\|_2 \mid x \in \mathcal{C} \right\} + 1$$

is well defined. Therefore, Taylor's formula (with (37) in mind) and assertion (i) of Lemma 1 yield

$$\begin{aligned} f(x^k + \delta d^k) &\le f(x^k) + \delta \nabla f(x^k)^\top d^k + \tfrac{1}{2}\delta^2 \theta \|d^k\|_2^2 \\ &\le f(x^k) + \delta(-\gamma + \tfrac{1}{2}\delta\theta)\|d^k\|_2^2 \end{aligned}$$

for all $\delta \in [0,\sigma]$ and all $k \in \mathbb{N}$. With $\delta_{\min} := \min\left\{\frac{\gamma}{\theta}, \sigma\right\} \in (0,\sigma] \subset [0,1]$, we have

$$-\gamma + \frac{1}{2}\delta_{\min}\theta \le -\frac{1}{2}\gamma$$

and

$$f(x^k + \delta_{\min} d^k) \leq f(x^k) - \frac{1}{2}\gamma\delta_{\min}\|d^k\|_2^2 \qquad \text{for all } k \in \mathbb{N}. \tag{38}$$

Therefore, taking into account assertion (v) of Lemma 1, the exact line search (20), and (38), it follows that

$$\left.\begin{array}{rcl} f(x^{k+1}) & = & f(\hat{x}^{k+1}) \\ & = & f(x^k + \delta_k d^k) \\ & \leq & f(x^k + \delta_{\min} d^k) \\ & \leq & f(x^k) - \frac{1}{2}\gamma\delta_{\min}\|d^k\|_2^2 \end{array}\right\} \quad \text{for all } k \in \mathbb{N}. \tag{39}$$

Since f is continuous on the compact set $\mathcal{C}$ and $\{x^k\} \subset \mathcal{C}$ due to (37), the sequence $\{f(x^k)\}$ is bounded below. Hence, (39) implies

$$\lim_{k\to\infty} d^k = 0.$$

Thus, Lemma 4 shows that $\bar{x}$ is a stationary point of problem (3) and, with this, a solution of EiCP. □

5. Numerical results

In this section we report some computational experience with the Spectral BAS algorithm on several sets of test problems from the literature. All experiments have been performed on a Pentium Intel® Core™ i7, 2.7 GHz, 16 GBytes of RAM memory and 64-bit operating system Windows®. The algorithm has been implemented in MATLAB® [23] environment (version 7.11, R2010b). The running times are always given in CPU seconds.

5.1. Test problems and implementation issues

We use five sets of test problems. In the first set, B is the identity matrix and $A \in \mathbb{R}^{n\times n}$ is a symmetric copositive matrix [12] related to the maximum clique problem [7]. For a simple and undirected graph $G = (V, E)$ with node set $V = \{1, \ldots, n\}$ and edge set E, a clique C is a subset of V such that every pair of nodes in C is connected by an edge in E. A maximum clique C is a clique with the maximum number of edges and its size $\omega(G)$ is called the (maximum) clique number [7]. Consider the symmetric matrix

$$A(\kappa) = \kappa(E_n - A_G) - E_n \tag{40}$$

where E_n is a matrix of order n whose elements are all equal to one and $A_G = (a_{ij})$ is the (symmetric) adjacency matrix of the graph G (i.e., $a_{ij} = 1$ if $\{i, j\} \in E$, and $a_{ij} = 0$ else, $i, j \in \{1, \ldots, n\}$). Then $A(\kappa)$ is copositive if $\kappa \geq \omega(G)$ [8, 36]. So, the maximum clique number problem is related with the problem of verifying whether a matrix of the form (40) is copositive. It is easy to see that a (symmetric or not) matrix A is copositive if and only if all complementary eigenvalues of EiCP (1), with B being the identity matrix, are nonnegative [3]. So, algorithms for dealing with the maximum clique problem may be based on the efficient computation of complementary eigenvalues of symmetric matrices of the form (40) for a finite number of real values κ. Due to this property, we decided to investigate whether the Spectral BAS algorithm is efficient for the solution of the EiCP with this class of matrices.

In Table 1, we list the characteristics of the graphs from the DIMACS [14] collection and the generated graphs (cf. [39]), where the number n of nodes gives the order of the examined matrices. In all of these test problems the copositive matrix $A = A(\kappa)$ is obtained by (40) with $\kappa = \omega(G)$.

In our second and third set of test problems, B is the identity matrix and A is a matrix from the Harwell-Boeing collection available at Matrix Market [24], where A is symmetric positive definite (SPD) and indefinite (IND), respectively. The fourth set of test problems uses nondiagonal SPD matrices A and B and the fifth test set contains problems with a nondiagonal SPD matrix B and an indefinite matrix A. In the last two test sets, A is a matrix from the Harwell-Boeing collection and B is a symmetric matrix ($B = I_n + CC^\top$), where I_n is the identity matrix and the elements of C are randomly drawn from a uniform distribution in the interval $[0, 1]$. Test problems were scaled according to the procedure described in [21], which improves the efficacy of the algorithms, particularly when they are applied to EiCPs with ill-conditioned matrices A or B. The characteristics of SPD and indefinite matrices from the Harwell-Boeing collection are presented in Table 2.

5.2. Preprocessing and initial point

In this section, we focus our attention on the initial point that should be chosen in order to improve the efficiency and efficacy of the Spectral BAS algorithm. As discussed in [20], the barycenter of the simplex is usually the most common initial point to be used by the SPG algorithm. Our experiments not reported in this paper show that the Spectral BAS algorithm fails for some instances when the barycenter is used as initial point. A vector of the canonical basis e^i, $i = 1, \ldots, n$, can be a valid alternative for such an initial point. It is very easy to see whether a canonical vector e^i is a solution of EiCP, as the following property holds.

Proposition 2. *The vector e^i of the canonical basis is a solution of the EiCP if and only if $r_i := \min\{a_{ii}b_{ji} - a_{ji}b_{ii} \mid j = 1, \ldots, n\} \geq 0$.*

Based on this property, we designed a preprocessing technique which first checks whether one of the canonical vectors e^i, $i = 1, \ldots, n$, is a solution of EiCP. If none of these n vectors is a solution of EiCP, then the initial point is the canonical vector e^s, where

$$s = \operatorname{argmax}\{r_i \mid i = 1, \ldots, n\}. \tag{41}$$

The motivation of this preprocessing technique is twofold. First, for some instances a solution of EiCP can be easily found without using an algorithm. Furthermore, if this is not the case the technique gives a rule (41) for choosing an initial canonical vector that seems to work well in practice. For example, for the test problems in Table 3, the preprocessing technique reduced failures of achieving a tolerance of $\varepsilon = 10^{-6}$ from 4 to 2 for the Spectral BAS algorithm and from 7 to 4 for the SPG algorithm.

5.3. Performance of Spectral BAS algorithm

Tests for the the Spectral BAS (SBAS in the tables) algorithm have been performed for $\beta = 10^{-i}$, with $i = 0, \ldots, 6$. Our experiments showed an improvement in the performance of the Spectral BAS algorithm with the reduction of the parameter β. The algorithm has usually the best performance for $\beta = 10^{-5}$ as it solves most of the test instances with the smallest number of iterations. So, we used $\beta = 10^{-5}$ for obtaining the numerical results of the Spectral BAS algorithm presented in the paper. For all of the test problems the values of $\eta_{\min}$ and $\eta_{\max}$ have been fixed to 10^{-6} and 10^{6}, respectively. Furthermore, we set $\eta_0 = 1$. The tolerance ε in the stopping criterion of the algorithms has been chosen as 10^{-6} and a more relaxed tolerance ($\varepsilon = 10^{-4}$) was tested when the algorithm fails to attain a stationary point within the limit of 1000 iterations. We use the notation (*) when the algorithm Spectral BAS (and also SPG) was only able to terminate satisfying the stopping criterion with the more relaxed tolerance $\varepsilon = 10^{-4}$.

As discussed in [20], the symmetric EiCP can be efficiently solved by using the SPG algorithm applied to the fractional quadratic program on the simplex (3). In order to have a better idea of the efficiency of the Spectral BAS algorithm, we also solved all the test problems by this SPG algorithm. In Table 1, we present results of algorithms Spectral BAS and SPG, to solve the problems in test set 1 with the initial point defined by the preprocessing technique, where IT is the total number of iterations, λ is the complementary eigenvalue computed, and T/IT is the CPU time (in seconds) required per iteration by the algorithms (the value "0" means that the time per iteration is below 10^{-5} seconds). None of these test problems was solved in the preprocessing phase. This means that none of the n canonical vectors is a solution of the EiCP. Computational results indicate that the Spectral BAS algorithm solved very efficiently and got accurate solutions for all the test problems of the first set (matrices from the maximum clique collection). As stated before, this very good performance of the Spectral BAS algorithm for solving the EiCP with this class of matrices may have important implications on the design of new algorithms for the computation of the maximum clique number by solving a finite number of EiCPs.

Table 1: Performance of SBAS and SPG algorithms for generated small instances [39] and larger instances from DIMACS collection [14].

				SBAS			SPG		
Graph	n	$\|E\|$	$\omega(G)$	It	λ	T/It	It	λ	T/It
c-fat14-1	14	52	6	19	5.14E+00	0	21	5.14E+00	0
Brock14	14	51	5	24	5.65E+00	0	25	5.65E+00	0
Brock16	16	59	5	19	7.48E+00	8.68E-04	20	7.48E+00	0
Brock18	18	78	5	19	8.20E+00	0	19	8.20E+00	0
Brock20	20	95	5	16	8.82E+00	0	18	8.82E+00	0
Morgen14	14	50	5	16	5.73E+00	0	18	5.73E+00	0
Morgen16	16	59	5	16	8.61E+00	1.04E-03	13	8.61E+00	0
Morgen18	18	60	5	12	1.09E+01	0	13	1.09E+01	0
Morgen20	20	67	5	12	1.24E+01	0	13	1.24E+01	0
Morgen22	22	68	5	11	1.55E+01	0	12	1.55E+01	0
Johnson6-2-4	15	45	3	10	6.00E+00	0	11	6.00E+00	0
Johnson6-4-4	15	45	3	10	6.00E+00	0	11	6.00E+00	0
Johnson7-2-4	21	105	3	2	6.00E+00	0	2	6.00E+00	0
Jagota14	14	31	6	11	9.83E+00	0	12	9.83E+00	0
Jagota16	16	57	8	11	1.04E+01	0	12	1.04E+01	0
Jagota18	18	84	10	12	1.09E+01	0	12	1.09E+01	0
sanchis14	14	50	5	20	5.65E+00	0	25	5.65E+00	0
sanchis16	16	50	5	15	8.68E+00	0	17	8.68E+00	0
sanchis18	18	50	5	14	1.07E+01	0	16	1.07E+01	0
sanchis20	20	50	5	15	1.17E+01	0	12	1.17E+01	0
sanchis22	22	50	5	12	1.33E+01	0	13	1.33E+01	0
Brock200-1	200	14834	21	11	4.51E+01	0	12	4.51E+01	0
Brock200-2	200	9876	12	9	9.28E+01	1.95E-03	11	9.28E+01	0
Brock200-3	200	12048	15	9	7.16E+01	0	10	7.16E+01	0
Brock200-4	200	13089	17	9	6.17E+01	1.95E-03	11	6.17E+01	0
c-fat200-1	200	1534	12	6	1.83E+02	3.13E-03	7	1.83E+02	0
c-fat200-2	200	3235	24	7	1.66E+02	2.60E-03	8	1.66E+02	0
c-fat200-5	200	8473	58	10	1.14E+02	0	12	1.14E+02	1.42E-03
Hamming6-2	64	1824	32	50	5.16E+00	0	27	5.16E+00	0
Hamming6-4	64	704	4	13	3.47E+01	0	12	3.47E+01	0
Hamming8-2	256	31616	128	40	7.06E+00	4.01E-04	87	7.06E+00	1.09E-03
Hamming8-4	256	20864	16	12	8.21E+01	1.42E-03	13	8.21E+01	1.30E-03
Johnson8-2-4	28	210	4	13	1.43E+01	0	15	1.43E+01	1.12E-03
Johnson8-4-4	70	1855	14	17	1.29E+01	9.77E-04	18	1.29E+01	0
Johnson16-2-4	120	5460	8	12	1.60E+01	0	13	1.60E+01	0
Johnson32-2-4	496	107880	16	11	3.20E+01	1.56E-03	23	3.20E+01	2.13E-03
Keller4	171	9435	11	13	5.13E+01	1.30E-03	15	5.13E+01	0
Mann-a9	45	918	16	15	3.20E+00	0	11	3.20E+00	1.56E-03
Mann-a27	378	70551	126	24	4.82E+00	6.79E-04	16	4.82E+00	2.08E-03

In contrast to the previous results, for the sets 2–5 of test problems, the preprocessing technique is able to find a solution of the EiCP in 89% of the instances (among them all instances of test set 4). In Table 3, we report the results obtained by the Spectral BAS and SPG algorithms for those instances of the test sets 2, 3 and 5, where the preprocessing technique was not able to find

Table 2: Characteristics of matrices from Harwell-Boeing collection.

SPD matrix	n	SPD Matrix	n	Indefinite Matrix	n
662_bus	662	bcsstm11	1473	494_bus	494
685_bus	685	bcsstm19	817	bcsstk19	817
1138_bus	1138	bcsstm20	485	bcsstk20	485
bcsstk01	48	bcsstm21	3600	bcsstk21	3600
bcsstk02	66	bcsstm22	138	bcsstk22	138
bcsstk03	112	bcsstm23	3134	bcsstk23	3134
bcsstk04	132	bcsstm24	3562	bcsstk24	3562
bcsstk05	153	bcsstm25	15439	bcsstk25	15439
bcsstk06	420	bcsstm26	1922	bcsstk26	1922
bcsstk07	420	gr_30_30	900	bcsstk28	4410
bcsstk08	1074	nos1	237	bcsstm27	1224
bcsstk09	1083	nos2	957	bfw398b	398
bcsstk10	1086	nos3	960	bfw62b	62
bcsstk11	1473	nos4	100	bfw782b	782
bcsstk12	1473	nos5	468	lund_a	147
bcsstk13	2003	nos6	675	lund_b	147
bcsstk14	1806	nos7	729	mhd3200b	3200
bcsstk15	3948	s1rmq4m1	5489	mhd4800b	4800
bcsstk17	10974	s1rmt3m1	5489	odep400b	400
bcsstk18	11948	s2rmq4m1	5489	plat1919	1919
bcsstk27	1224	s2rmt3m1	5489	plat362	362
bcsstm02	66	s3rmq4m1	5489	zenios	2873
bcsstm06	420	s3rmt3m1	5489		
bcsstm08	1074	s3rmt3m3	5357		
bcsstm09	1083				

Table 3: Performance of SBAS and SPG algorithms for test sets 2, 3, and 5.

	SBAS		SPG	
Matrix A	It	T/It	It	T/It
SPD	$B = I_n$			
bcsstk02	62	2.56E-04	48	3.32E-04
bcsstk04	97	1.63E-04	91	6.94E-04
bcsstk05	20	8.22E-04	836	6.18E-04
bcsstk10	449	2.79E-04	525*	4.32E-03
bcsstk27	192	2.44E-04	31*	5.21E-03
s1rmq4m1	403	4.00E-03	484	5.06E-02
s1rmt3m1	61*	4.43E-03	379	5.03E-02
s2rmq4m1	388	2.10E-03	98*	5.36E-02
IND	$B = I_n$			
bcsstk28	33	3.96E-02	227	6.77E-02
bcsstm27	460	2.42E-03	198*	7.46E-03
lund_a	80	1.98E-04	112	5.63E-04
lund_b	39*	4.11E-04	137	6.89E-04
IND	$B = I_n + CC^\top$			
bfw398b	32	1.01E-03	35	1.84E-03
bfw62b	38	1.69E-03	39	1.64E-03
bfw782b	30	2.16E-03	33	4.88E-03

a solution of the EiCP. The Spectral BAS algorithm solved efficiently the test problems of the set 5, where A is a matrix from the Harwell-Boeing collection and B is a SPD matrix different from the identity. It seems that the Spectral BAS algorithm faces more difficulties to find a solution when A is still a matrix of the same collection and B is the identity matrix. For these instances, the Spectral BAS algorithm was able to solve all the problems for a relaxed stopping criterion ($\varepsilon = 10^{-4}$), but fails to terminate twice when a more accurate solution is needed ($\varepsilon = 10^{-6}$). The results show that the Spectral BAS algorithm is competitive in general with the SPG method, as this latter algorithm was able to solve efficiently all the test problems of the sets 1 and 5 and of the sets 2 and 3 when the accuracy of the solution is not too demanding ($\varepsilon = 10^{-4}$). However, SPG algorithm has four failures if a higher accuracy of the solution is required ($\varepsilon = 10^{-6}$). Furthermore, time per iteration T/It is smaller for the Spectral BAS algorithm and this gap tends to increase with the dimension of the EiCP. This is mainly due to the fact that the Spectral BAS algorithm uses in each iteration a projection on $\mathbb{R}^n_+$ while a projection on the simplex has to be performed in each iteration of the SPG algorithm.

To give a more illustrative comparison of the performances of Spectral BAS and SPG algorithms we report in Figures 1 and 2 the Dolan–Moré performance profiles [15] of the two algorithms based respectively, on the number of iterations and the total CPU time for test problems of sets 1 to 5 for which the preprocessing technique was not able to find a solution of EiCP. We see that the Spectral BAS algorithm performs better on the test set than SPG, in terms of both efficiency and robustness. In fact, reading the values of the curves of Figure 1 for a factor $\tau = 1$, we can observe that Spectral BAS is able to attain the best metric value for about 80% of the problems. In terms of robustness, and reading the same curves for large values of τ, we observe that Spectral BAS successfully solved more than 90% of the problems. For the performance profile based on CPU time (Figure 2) we left out those examples for which both the Spectral BAS and the SPG algorithm needed less than 10^{-3} seconds since the differences of times are meaningless.

In addition, we investigated the impact of the parameter β used in the construction of the active and inactive sets of Spectral BAS algorithm. For

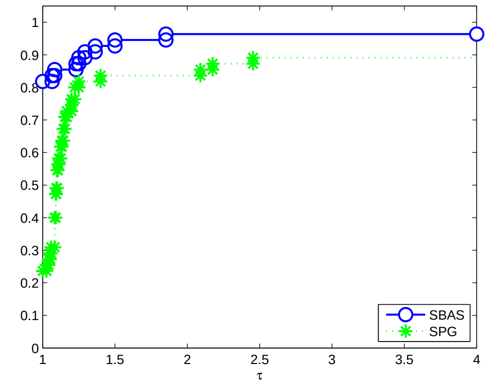

Figure 1: Comparing SBAS and SPG based on performance profiles of the iterations.

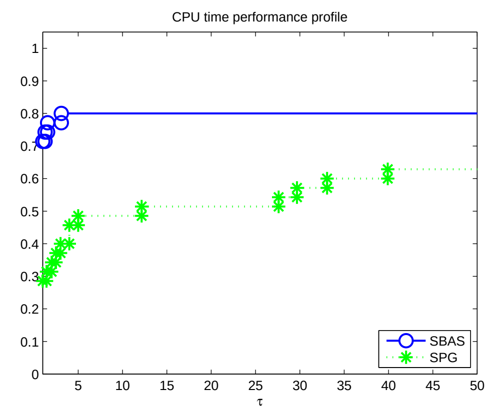

Figure 2: Comparing SBAS and SPG based on performance profiles of the total CPU time.

instance, for $\beta = 10^{-3}$ the Spectral BAS algorithm with the initial point given by the preprocessing technique (41) needs 263 iterations to find a solution of the EiCP for problem `s1rmt3m1` and 80 iterations to find a solution of problem `lund_b` if $\beta = 7 \times 10^{-1}$. Since the Spectral BAS algorithm is very fast, we recommend to use $\beta = 10^{-5}$ initially and use a different set of values for β whenever the Spectral BAS algorithm faces difficulties to terminate in a reasonable number of iterations.

In recent years much attention has been given to techniques for forecasting active sets in the context of (quadratic) l_1-regularized optimization (see e.g. [35, 13]) or convex quadratic problems with nonnegativity constraints [11]. The design of a more elaborated strategy based on some of these ideas or others for the choice of β is surely an important issue for future research.

6. Conclusions

In this paper, a new Spectral Block Active Set algorithm is proposed to deal with the symmetric Eigenvalue Complementarity Problem (EiCP). The algorithm seeks a stationary point of an equivalent fractional quadratic program on the simplex and seems particularly recommended for large-scale EiCPs. Global convergence for the algorithm is established. Numerical experience is reported illustrating that the algorithm is quite efficient for the solution of the EiCP (1) when B is the identity matrix and A is a modification of the adjacency matrix of a graph associated to the maximum clique problem. This performance may have important implications on the solution of this difficult problem. When A is a general matrix and B is still the identity matrix, the Spectral BAS algorithm is very efficient to find a solution that is not very accurate but may have difficulties when a more accurate solution is needed. The same problem was observed for the SPG method, but in more cases. The Spectral BAS algorithm seems to be competitive to the SPG method that was proposed in [20] for the solution of the symmetric EiCP by dealing with the same fractional quadratic program. However, the computational effort per iteration is usually smaller for the Spectral BAS algorithm as this last method employs simpler projections. Numerical experiments also indicate that the constant β used in the technique for forecasting the active-set in each iteration and the choice of the initial point are two important issues for the Spectral BAS to be more efficient. These points were discussed in some detail in this paper and should be investigated in the near future. The use of ideas similar to those of the Spectral BAS algorithm for the solution of the Quadratic Eigenvalue Complementarity Problem [10, 32] and the Second-Order Complementarity Eigenvalue Problem [1, 17, 33] should also deserve attention in our future research.

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